# Endogenous Switching of Volatility Regimes: Rational Expectations and Behavioral Sunspots*

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March 20, 2009

**Abstract**

Recent trends in macroeconomics focus on the importance of central bank communication policies (Morris and Shin, 2002) and statistical learning (Evans and Honkapohja, 2001) in expectations formation. The present work merges and innovates basic ideas from both approaches. Firstly, we analyse a Lucas-type monetary model where private sector expectations are polarized by two, and not only one, institutional forecasters. Rating agencies, market leaders, fiscal authorities generally influence private sector expectations as well, and sometimes more, than the central bank. When this is the case, strategic motives arise because the rational expectations equilibrium results as solution of a simultaneous coordination game among such big actors. Therefore, and this is a second novelty, both institutional forecasters use constant gain adaptive learning not only to learn about fundamentals but also to assess rationality of the other institutional forecaster's expectations. Specifically, we assume each institutional forecaster doesn't have perfect information about the other one's simultaneous expectations, but only a noisy signal of it. Because their non negligible effect on aggregate expectation, they both do have a private incentive to learn how to weight these signals in order to improve their own forecasting performance. Institutional polarization and the incentive to learn about others' rationality shape a twofold expectations interdependency. It is proved how the stochastic nature of this economy can give rise to endogenous, unpredictable and persistent switches in volatility regimes. Specifically, inflation dynamics can suddenly jump from the unique rational expectation equilibrium to a behavioral sunspot equilibrium and viceversa. The latter entails an excess volatility regime generated by institutional forecasters using the noisy signals to form their expectations. The non-neutrality of the information diffusion channel from institutional forecasters to the private sector plays a key role in triggering this new equilibrium. Numerical simulations give a quantitative insights of such findings.

**JEL codes:** C62; D83; D84; E40.

**Keywords:** constant gain adaptive learning, expectations co-movements, expectations polarization.

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# 1 Introduction

## 1.1 Changes in volatility regimes

This paper aims to provide a stylized model on how unpredictable and endogenous changes of volatility regimes can arise mainly because agents fail to form expectations independently. Volatility is one among the most important sources of uncertainty. Generally the higher and more frequent are fluctuations in the economy the higher are costs paid in terms of insurance or financial fragility. One of the most challenging task for economists is to understand when and how an high volatility crisis triggers.

The issue of excess volatility has recently received attention by the profession with special regard for US time series evolution after the second word war onwards (figure below).

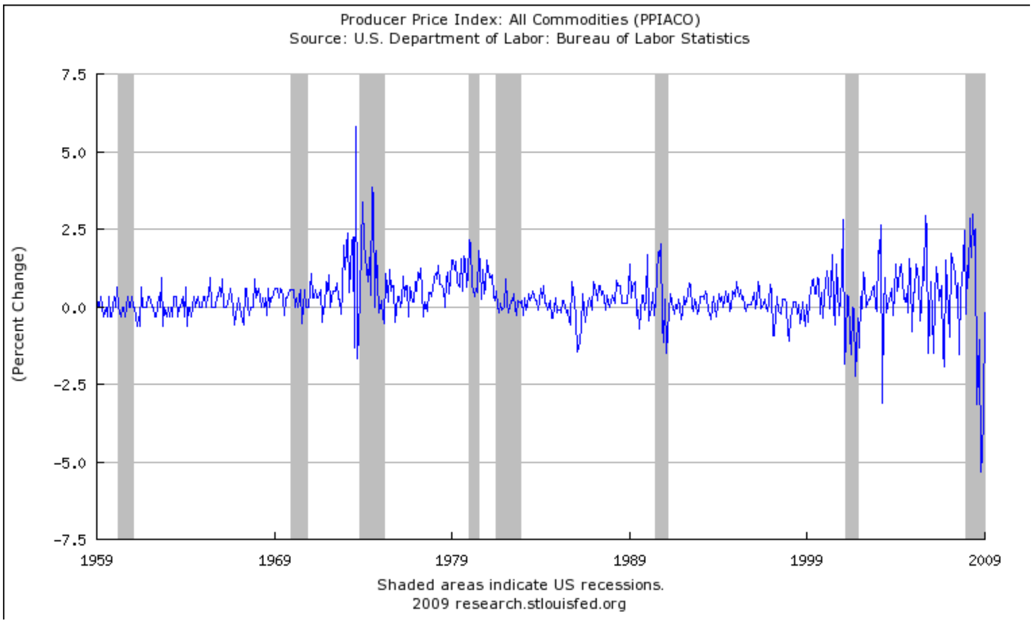


Figure 1: US inflation (percentage change of Production Prices Index) time series of last fifty years. Grey bars denote recession periods. The picture suggests different volatility regimes with no strict correlation between fluctuation amplitude and growth cycles.

Cogley and Sargent (2005a), Sims and Zha (2006), Primiceri (2005) have found several different drifting and volatility regimes. These studies seems to give little importance to quantitative effect of monetary policy. Sims and Zha (2006) notice that: *"the work of Cogley and Sargent and Primiceri all fits with the notion that the data do not deliver clear evidence of parameter change unless one imposes strong, and potentially controversial, overidentifying assumptions"*. Paying a such price, Cogley and Sargent (2005b) and Primiceri (2006) gives an explanation volatility changes in terms of learning by the central bank. In their study there is evidence that high inflation in 70's would had risen because some years passed for the FED to correctly identify the model. In other words, Lucas' lesson would have been learned after evidence of it has been produced by the implementation of wrong policies. Nevertheless, even if partially in conflict, all these findings are in line with the suggestion coming from the picture above that high volatility periods and recessions are not significantly correlated. This would address the issue of volatility inflation changes to change in components not so strictly linked to cyclical real economy determinants.

## 1.2 Behavioral uncertainty, interdependent expectations and learning

An important part of the profession places now more and more emphasis on the role of central bank as focal point for agents' expectations, stressing the pre-eminence of the communication policy on the mere control of monetary determinants of the economy (a good introduction to the issue is Morris and Shin (2007)). Agents look at central bank expectations because everyone knows all others are looking at it, so that, central bank expectations provides noisy information on what the others are simultaneously expecting. This simple empirical fact tells economic theory that the idea individuals are able to hold rational expectations *independently* cannot hold. If this was the case, each one would simply have all relevant information and signals coming from the central bank would be just redundant.

Adaptive learning (Marcet and Sargent 1989, Evans and Honkapohja 2001) answers the need to design a more reasonable and dynamic theory of expectation formation in contrast with dogmatic acceptance of rational expectations hypothesis. The central idea is that agents act as econometricians. They form expectation according to a theory (a perceived law of motion) calibrated estimating recursively the impact of relevant variables as data become available in real time. This more realistic way to think about expectations as a further dimension of the dynamics of the system introduces new issues as the use of misspecificated theories (Evans, Honkapohja and Sargent 1989, Sargent 1999, Evans and Honkapohja 2001) and evolutionary competition among alternative statistical predictors (Branch and Evans 2006, 2007, Guse 2005).

In this paper, both basic ideas, namely the one about the importance of the coordination role of central bank and the one about adaptive formation of expectations, are merged[1] and innovated in two respects to explain changes in volatility regimes. We analyse the setting in which, firstly, more than one, in this case two, institutional forecasters polarize private sector expectations and, secondly, professional forecasters use adaptive learning not only to learn about fundamentals but also to assess rationality of other agents' expectations.

Rating agencies, market leaders, fiscal authorities generally influence private sector expectations as well, and sometimes more, than the central bank. In general, whenever more than one agent has non-negligible impact on aggregate expectations, holding rational expectations is a best action if and only if all others do the same. Therefore, because behavioral uncertainty, agents have the incentive to understand how others' expectations affect the actual economic course. The interaction entailed by the coordination expectations game among institutional

[1] Both arguments could have a further point of contact in the idea that estimation is a costly activity. The most part of agents cannot solve individually their own forecasting problem because the great amount of resources (at least cognitive) needed in gathering and processing all information to "produce" statistically consistent prediction. Therefore, agents look at central bank that maintains sufficient resources to form expectations according, in the best of cases, to an optimal statistical analysis of available data in light of the right theory underlying the working of the economy.

forecasters is a first source of expectations interdependence. A second one is entailed by the role of institutional forecasters acting as focal point for private sector's expectations. The latter is a one-way dependence linking private sector's beliefs to institutional forecasters' expectations whereas the former is a reciprocal interdependence among institutional forecasters' expectations.

In this paper, particular emphasis is placed on interaction between constant gain adaptive learning and these two forms of expectations interdependence being among primary reasons for persistent excess volatility triggering. To the aim we will simplify the setting in order to enlighten the basic mechanism and to make the analysis rigorous but handy. We don't want to neglect other determinants of excess volatility, but we aim to covey a first idea on how behavioral uncertainty alone can be enough to generate endogenous and unpredictable switches in volatility regimes.

### 1.3 Learning and communication

This work is a natural extension of Gaballo (2009). That paper investigates the learning dynamics of two institutional forecasters affected by behavioral uncertainty, but perfectly informed about both the exogenous determinants and the self-referential nature of the economy. Behavioral uncertainty arises in the sense they only have noisy observations of the simultaneous expectation of the other agent. Each agent estimates a coefficient weighting the noisy signal in expectation formation process responding to her own incentive to refine their forecasts. An equilibrium requires expectations to be locally optimal linear projection given behavioral uncertainty restrictions on the information set. Rational expectations equilibrium occurs whenever the estimated coefficient is zero, so that the behavioral noisy information is discarded. Otherwise, because the endogenous and interactive working of the learning algorithm, some equilibria different from rational expectations equilibria can arise entailing excess volatility regime. Those equilibria have been tagged Behavioral Sunspots Equilibria (BSE). They configure as a coordination failure in that both agents use irrelevant information.

In the present study, this scheme governs the arising of endogenous interdependence among institutional forecasters' expectations and it is implemented in a simple monetary Lucas-type model (the same of Evans and Branch (2006)). The microfoundation of the model is presented in the first section, nevertheless the results are not linked to the particular specification at hand. There are no novelties in the model in se. The setting is very simple but it obeys to general economic incentives and constraints such as transversality conditions and non negativity of prices. This is enough to deal with one among the primary concerns of this paper, that is, to defend, in principle, economic relevance of BSE.

Four new directions are explored. First, institutional forecasters learn not only about others' rationality but *also* about the fundamentals. This way it is showed how to merge the theme of learning about others' rationality with the classical theme of learning about fundamentals already developed in standard adaptive learning literature. We will refer to these two connected learning dynamics as the learning determinants of inflation dynamics. Second, the transmission channel from institutional forecasters to private sector is not neutral in that institutional forecasters' estimates are imitated with a noisy, possibly correlated, perturbation. This feature adds a truly macroeconomic flavour in that it reconciles the classical "forecasting the forecast of others' problem", where agents have non negligible impact on aggregate outcomes, with a non-trivial general equilibrium perspective, where an ocean of negligible agents are assumed. In other words, the problem faced by institutional forecasters is not their mere expectations coordination problem because the non-neutrality of the information transmission channel form them to the ocean of agents forming the private sector possibly alters the feedback mechanism. This constitutes what we will call the communication determinant of inflation dynamics. Third, institutional forecasters use a constant gain learning algorithm instead of recursive ordinary least square. Differently from recursive ordinary least square this rule is time invariant and it allows for persistent learning. In this work we will show an example on how constant gain can not only learn a structural change, but it can also trigger endogenously it. Finally, the paper extends analysis of BSE learneability to the case of correlation between institutional forecasters' observational errors. This is a natural extension since institutional forecasters are part of the same public environment, so that they are influenced by same factors. In other word, it is likely in some extent that either both have a pessimistic perception of the other's expectations or both have an optimistic one.

### 1.4 Switching from REE to BSE and viceversa

The main result of the paper is to provide a simple model that exhibit very standard rational expectations behavior *and,* at the same time, it has the potentiality to trigger persistent and endogenous changes in volatility without relying on any Markov switching or additional aggregate shock. All this is basically due to the endogenization of agents' beliefs coordination as described in Gaballo (2009). Here the basic mechanism is implemented and further developed in the context of a simple monetary model in order to defend, in principle, economic relevance of BSE.

The rational expectations equilibrium learneability is proved to generally hold and to be particularly robust to correlation in observational errors. Nevertheless, *even if* institutional forecasters successfully learn about economic fundamentals and rationality of others, one learnable BSE may suddenly arise. Conditions for emergence of a learneable BSE are fulfilled if the transmission channel from institutional forecaster to private sector causes even a very little average amplification of the signal passed by institutional forecasters. The striking feature of this model is that BSE is not alternative to rational expectations equilibria, but they coexist in a large region of the parameter space. In such a region, real time constant gain learning dynamics selects among them and endogenous and unpredictable switches from one equilibrium to the other can generally arise. I show with numerical simulations how a structural switch from the rational expectations equilibrium to BSE may occur endogenously. The intuition for the result is the following. Deviations of at least one institutional forecaster from the rational expectation results in displacements of aggregate expectation from the rational one. Persistent deviations from rational expectations equilibria results in this model because, even if agents are all rational and able to consistently estimate fundamentals, they may fail to extract the signal of others' rationality, falling into a coordination failure trap.

### 1.5 Related literature on excess volatility

Branch and Evans (2007) reconsider the theme of learning but they shifts attention on expectation formation among all economic operators in a simple self referential model. Even if central bank is typically the most authoritative forecasting institution several different theories of the same economy are actually employed by agents to forecast. In an evolutionary contest competing theories can coexist because no one is able to perform better than others given their distribution over the population. Using a Lucas-type monetary model, Branch and Evans shape such environment in which different underparameterized theories are available to agents that chose among them on the basis of past performance. They show that Misspecification Equilibria (Branch and Evans 2006a) can arise giving rise to persistent stochastic volatility. Notice stochastic volatility is permanent and finally relies on the unavailability of a correctly specified predictor, the only one potentially consistent with REE.

Related is also an extensive stream of literature on excess volatility in asset market returns. We can distinguish mainly four approaches in macroeconomics. First, Timmermann (1993, 1996), Brennan and Xia (2001) and Cogley and Sargent (2006) among others assume agents implementing Bayesian learning on the dividend process. Those models are not self-referential nature, since agents beliefs do not influence the market outcomes. It is common sense and a simple empirical exercise to test that financial operators do react to changes in prices as well. Differently, Carceles-Poveda and Giannitsarou (2008), Adam, Marcet and Nicolini (2008a) and Bullard, Evans and Honkapohja (2007) properly takes in to account agents adaptively learning about the prices level. As clarified by Adam, Marcet and Nicolini (2008b) learning about the price level is justified by uncertainty on the marginal agents' expectations, therefore, this scheme considers implicitly the self-referential nature of the model. Later works building on Brock and Hommes (1997, 1998) assumes agents choose among a set of very less sophisticated predictors of the price level relying on relative past performance. Such setting can give rise to complex dynamics and strange attractors. Finally, a recent approach initiated by Allen, Morris and Shin (2006) focuses on the role of high order beliefs of rational short lived agents.

The most important feature of the proposed model in front of quoted literature is that the model is consistent and not alternative to REE. In other words, models above rely on some mechanism that either is exogenously imposed at an aggregate level or persistently alters the volatility regime of the dynamics. Differently, persistent high volatility regimes endogenously (and unpredictably) arise in this model from a REE regime and viceversa. Moreover the extra noise possibly entering in the equilibrium solution is justified at a micro level, that is, it is not an arbitrary aggregate shock.

# 2 Model

## 2.1 A Lucas-type economy

The primary concern of this section is to provide a simple fully microfounded model in order to defend, in principle, the economic relevance of behavioral sunspot equilibria. Of course the choice is functional to the aim, so the model is rich enough to embodies standard economic incentives and constraints usually assumed, but also simple enough to have a handy reduced form. Specifically, we will derive a simple Lucas-type monetary model where expectations of current inflation influences actual inflation. It is not a task of this paper to introduce novelties concerning the model in se. To make easier the comparison of this paper with closest literature, we will assume the same model with slightly different notation as in Branch and Evans (2007). The key assumptions are the following. We use the convention of a yeoman farmer model (as in Woodford (2003)) provided with a money-in-the-utility function. This is enough to generate a non trivial demand for money responding to classical quantity theory of money (see Walsh (2003)) without referring to any specification of the financial market. Finally, we assume a fraction of firms have to set quantities a period before. The latter yields expectations about current inflation to matter. We now detail the model.

**Households.** Each farmer produces a differentiated good and sells it in a monopolistically competitive market. In order to introduce price stickiness it is enough to allow for endogenous goods supply. Technology for a representative firm belonging to industry $i$ is given by the following

$$Y_{it} = \psi_t \Omega_t^{-1/(1+\eta)} N_{it}$$

where $\Omega_{t-1}$ is the unit labor requirement, $\psi_t$ is a stochastic disturbance and $N_{it}$ is the quantity of labour specific for industry $i$ employed at time $t$. Let's assume two type of industries; extension to an arbitrary number is straightforward. A representative households solves

$$\max_{\{C_{it},M_{it},N_{it},B_{it}\}} E_0^i \sum_{t=0}^{\infty} \beta^t \frac{C_{it}^{1-\gamma} + M_{it}^{1-\gamma}}{1-\gamma} - \frac{N_{it}^{1+\eta}}{1+\eta}$$

$$\text{s.t.} \quad C_{it} + M_{it} + B_{it} = Y_{it} + \frac{P_{t-1}}{P_t} M_{it-1} + \frac{P_{t-1}}{P_t} \left(1 + i_{t-1}\right) B_{it-1}$$

where $i_t$ is the nominal one-period interest rate on debt, $E^i$ is conditional expectation given agent $i$'s information set and

$$\begin{aligned} C_i &= \left( \int C_{i,j}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \\ P &= \left( \int P_j^{1-\theta} dj \right)^{\frac{1}{1-\theta}}, \end{aligned}$$

are CES indexes with $C_{i,j}$ and $P_{t,j}$ being respectively consumption of good $j$ by agent $i$ and price of good $j$. The aggregate demand $Y_t$ is equal to the integral of individual cost-minimizing demand over agents and goods, formally

$$Y_t = \int\int \left(\frac{P_{j,t}}{P_t}\right)^{-\theta} n_i C_{it}\ di\ dj = \int Y_{it}\ di = \int \left(\frac{P_{j,t}}{P_t}\right)^{-\theta} Y_{j,t}\ dj$$

where $Y_{it}, Y_{jt}$ and $n_i$ are respectively individual aggregate demand over goods $j$, aggregate demand of good $j$ over individuals $i$ and the fraction of firm type $i$. The household's first-order conditions can be written as,

$$\begin{aligned}
C &: \beta^t C_{it}^{-\gamma} - \lambda_{it} = 0 \\
M &: \beta^t M_{it}^{-\gamma} - \lambda_{it} + E_t^i \lambda_{it+1} \frac{P_t}{P_{t+1}} = 0 \\
B &: -\lambda_{it} + (1 + i_t)\, E_t^i \lambda_{it+1} \frac{P_t}{P_{t+1}} = 0 \\
N &: -\beta^t N_{it}^{\eta} + \lambda_{it} \frac{Y_{it}}{N_{it}} = 0
\end{aligned}$$

where $\lambda_{it}$ is the Lagrangian multiplier for the budget constraint. We can rewrite condition above solving for $\lambda_{it}$. We obtain

$$\begin{aligned}
C_{it}^{-\gamma} &= \Omega_{it-1} Y_{it}^{\eta}, \\
C_{it}^{-\gamma} &= M_{it}^{-\gamma} + \beta E_t^i C_{it+1}^{-\gamma} \frac{P_t}{P_{t+1}} \\
C_{it}^{-\gamma} &= \beta(1 + i_t) E_t^i C_{it+1}^{-\gamma} \frac{P_t}{P_{t+1}}.
\end{aligned}$$

These conditions must be satisfied for all $i$ and in all $t$ . In the steady-state $P_{t+1}/P_t = 1$ and $\beta(1 + i_t) = 1$. Combining Euler equations above it is possible to solve for the money-demand function:

$$M_{it} = \left(\frac{i_t}{1 + i_t}\right)^{-\frac{1}{\gamma}} C_{it}.$$

Following Walsh, and letting $\gamma \to \infty$, equilibrium in the money-market requires

$$M_t = Y_t$$

and taking logs of both sides we derive a simple version of the well-known quantity theory of money

$$\ln \overline{M}_t - \ln P_t = \ln Y_t \ , \tag{1}$$

where $\overline{M}_t$ is real money supply. The latter represents the aggregate demand (AD) equation. Notice also such assumption makes consumption and money demand to be independent from (heterogeneous) expectations on future inflation rate. The latter represents the aggregate demand (AD) equation.

**Production.** Firms set price to maximize profits. Let $P_{i,t}$ be the price in industry $i$ settled by firms taking as given the aggregate price-index $P_t$. Then a firm's profit function is

$$\Pi \equiv (P_{i,t} - P_t)\, Y_{i,t} = P_{i,t} Y_{i,t} - \frac{\psi_t \Omega_{t-1} Y_{i,t}^{1+\eta} P_t}{C_{i,t}^{-\gamma}}.$$

whose F.O.C. is,

$$\frac{\partial \Pi}{\partial P_{i,t}} + \frac{\partial \Pi}{\partial Y_{i,t}} \frac{\partial Y_{i,t}}{\partial P_{i,t}} = Y_{i,t} + \left( -\theta Y_t \left( \frac{P_t}{P_{i,t}} \right)^{\theta} \right) \left( P_{i,t} - \frac{\psi_t \Omega_{t-1} (1+\eta)\, Y_{i,t}^{\eta} P_t}{C_{i,t}^{-\gamma}} \right) = 0$$

that reduces to

$$\left( \frac{P_{i,t}}{P_t} \right)^{1+\theta\eta} = \frac{\theta}{\theta - 1} \frac{\psi_t \Omega_{t-1} Y_t^{\eta}}{C_{i,t}^{-\gamma}},$$

or, in log form

$$\ln (P_{i,t}) = \ln(P_t) + \frac{\eta}{1+\theta\eta} \ln(Y_t) - \frac{\gamma}{1+\theta\eta} \ln(C_{i,t}) + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \ln \left( \frac{\theta}{\theta - 1} \psi_t \right).$$

Following Woodford, assume there is a fraction $\tau$ of firms that set prices optimally in every period, while the remaining set their prices one period in advance. Denote $P_{i,f}$ ,$P_{i,d}$ as the prices of an industry $i$ respectively of type $f$ with flexible prices and $d$ with predetermined prices. Then the log-linearized pricing equations are:

$$\begin{aligned} \ln P_{i,f,t} &= \ln(P_t) + \frac{\eta}{1+\theta\eta} \ln(Y_t) - \frac{\gamma}{1+\theta\eta} \ln(C_{i,t}) + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \zeta_t \ , \\ \ln P_{i,d,t} &= E_{t-1}^{d} \ln P_{i,f,t} \ . \end{aligned}$$

where $\zeta_t$ collects the stochastic term in $\psi_t$. With all agents types evenly distributed across industries it follows that the aggregate price-index can be approximated as,

$$\ln P_t = \tau \left( n \ln P_{1ft} + (1-n) \ln P_{2ft} \right) + (1-\tau) \left( n E_{t-1}^{1} \ln P_{1ft} + (1-n)\, E_{t-1}^{2} \ln P_{2ft} \right)$$

and since[2] $E_{t-1}^{i} \ln P_{ft} = E_{t-1}^{i} \ln P_t$ we have

$$\begin{aligned} \ln P_t - E_{t-1} \ln P_t &= \frac{\tau}{1-\tau} \left( n \ln P_{1ft} + (1-n) \ln P_{2ft} - \ln P_t \right) \\ &= \frac{\tau}{1-\tau} \left( \left( \frac{\eta - \gamma}{\theta} \right) \ln Y_t + \frac{1}{1+\theta\eta} \ln \Omega_{t-1} + \zeta_t \right) \end{aligned}$$

Therefore, we have the aggregate supply (AS) relation

[2] Consider $E_{t-1}^{i} \ln p_t = \tau E_{t-1}^{i} \ln p_{ft} + (1-\tau)\, E_{t-1}^{i} \ln p_{ft}$ .

$$q_t \equiv \ln Y_t - \ln \Omega_{t-1} = \varphi_1 \left(\ln P_t - E_{t-1} \ln P_t\right) + \varphi_2 \ln \Omega_{t-1} + \varphi_3 \zeta_t$$

with

$$\varphi_1 = \frac{\theta (1-\tau)}{\tau (\eta - \gamma)}, \quad \varphi_2 = -\frac{1+\tau\theta\eta}{\tau + \tau\theta\eta}, \quad \varphi_3 = -\frac{(1-\tau)}{\tau},$$

where, for example, $\ln \Omega_{t-1}$ follows a deterministic trend. The AS is a kind of new classical Phillips curve encompassing the one in Lucas (1973), Kydland and Prescott (1977), Sargent (1999), and Woodford (2003).

The economy is represented by equations for aggregate supply (AS) and aggregate demand (AD):

$$\begin{aligned} AS \quad &: \quad q_t = \varphi_1 (p_t - p_t^e) + \varphi_2 \omega_{t-1} + \varphi_3 \zeta_t \\ AD \quad &: \quad q_t = m_t - p_t \end{aligned}$$

where $p_t$ is the log of the price level, $p_t^e$ is the log of expected price formed in $t-1$, $m_t$ is the log of the money supply, $q_t$ is the deviation of the log of real GDP from trend, $\zeta_t$ is an i.i.d. zero-mean shock, and $\omega_t$ is the log of the unit labor requirement.

**Monetary Authority.** Assume that the money supply follows

$$m_t - p_t = -(1+\xi)\left(p_t - p_{t-1}\right) + \delta \omega_{t-1} + u_t \quad \text{with} \quad \xi \geq 0,$$

where $u_t$ is a white noise money supply shock. We are assuming that central bank can observe both $p_t$ and $y_t$ as in Sargent (1987) and Evans and Ramey (2006). Denoting $\pi_t = p_t - p_{t-1}$ we can write the law of motion for the economy in its expectations augmented Phillips curve form

$$\pi_t = \frac{\varphi_1}{1+\varphi_1+\xi} \pi_t^e + \frac{\varphi_2 - \delta}{1+\varphi_1+\xi} \omega_{t-1} + \frac{\varphi_3}{1+\varphi_1+\xi} \zeta_t - u_t$$

or

$$\pi_t = \alpha \omega_{t-1} + \beta \pi_t^e + \nu_t \tag{2}$$

where $\beta = \frac{\varphi}{1+\varphi+\xi}$, $\alpha = \frac{\varphi_2 - \delta}{1+\varphi+\xi}$, $\nu_t \equiv \frac{\varphi_3}{1+\varphi_1+\xi} \zeta_t - u_t$. Note, in particular, that $0 \leq \beta < 1$. The reduced form of this Lucas-type model is very close to the Cobweb one. The difference between the two is in the sign of the feedback from expectations: the latter entails a negative feedback, the former a positive one. Differently from the new-Keynesian framework, inflation at time $t$ is affected by expectations at time $t-1$ instead that simultaneous expectations.

**Equilibrium.** A rational expectations equilibrium (REE) is a stationary sequence $\{\pi_t\}$ which is a solution to (2) given $\pi_t^e = E_{t-1}\pi_t$ , where $E_t$ is the conditional expectations operator. It is well known that (2) has a unique REE and that it is of the form

$$\overline{\pi}_t = \frac{\alpha}{1-\beta} \omega_{t-1} + \nu_t. \tag{3}$$

The REE is a stationary process and cannot explain empirically relevant volatility switching. This paper will deal with such unsatisfactory property modifying expectation formation process. As it will be clear later, the reinforcement effect of agents' expectations is a necessary but not sufficient feature for our purposes. Specifically, provided the expectations feedback effect is always damping, the information transmission in the economy will play a key role for emergence of endogenous volatility regime switching. The following section will detail how we are going to modify the rational expectations hypothesis.

## 2.2 Expectations formation and information diffusion

This section aims to describe how aggregate expectation forms and evolves in time. We will introduce two essential hypothesis in place of the rational expectation hypothesis. First, non-trivial behavioral uncertainty is in place, second, private sector expectations are polarized by two institutional forecasters.

**Non-trivial behavioral uncertainty.** A general way to model behavioral uncertainty is described in Gaballo (2009). It requires agents suffer a measurement error in detecting others' simultaneous expectations. Let's denote by $E_{t-1}^{i}(\cdot)$ agent $i$'s expectation on $(\cdot)$ at time $t-1$. Behavioral uncertainty is entailed formally by

$$\left(E_{t-1}^{i}E_{t-1}^{j}\pi_t - E_{t-1}^{j}\pi_t\right) \equiv v_{i,t-1} \sim \Upsilon\left(0,\delta\right) \tag{4}$$

where $\nu_{i,t-1}$ is a stochastic measurement error drawn from a generic centred distribution function $\Upsilon_i\left(0,\delta\right)$ with zero mean and finite variance $\delta$. In words, agent perceive noisily others' expectation. This is a first (reasonable) departure from REE paradigm in that, form a strict microfounded point of view, REE holds given common knowledge of every agent holds rational expectation. One may want to keep also non-centred distribution, or different type of distributions over the population. This has a sense and it can generate interesting dynamics, nevertheless it doesn't add nothing substantial to the aim of this paper but some complexity. Therefore we will strictly focus only on a unique centred distribution equal for every agent.

Behavioral uncertainty about others' expectations is at the basis of the "forecasting the forecast of others problem" originally posed by Townsend (1983). A following stream of literature investigates how agents can coordinate on rational expectations from a such disequilibrium initial conditions (Marcet and Sargent (1989), Sargent (1991), Singleton (1987), Kasa (2000) and Pearlman and Sargent (2004)). All these works consider explicitly a finite number of agents who form expectations independently.

Nevertheless, in a general equilibrium perspective, to which the concept of REE refers to, the behavioral uncertainty problem can be just trivial as long as independent idiosyncratic deviations from the rational expectation vanish in the aggregation of an infinite number of agents. In other words, as long as individual deviations from the rational expectation are truly random and population is large enough, behavioral uncertainty doesn't add nothing substantial to the individual forecasting problem. In this sense, behavioral uncertainty is non-trivial as long as agents' deviations from REE prescription are driven by a common factor whose identification is crucial to optimally solve the individual forecasting problem. In fact, a reasonable doubt that there could exist a non trivial part of agents deviating in a correlated way from REE prescriptions would in turn justify an individual rational departure from REE prescription.

Typically, the coordination of expectations on a particular deviation is yield by the introduction of an exogenous variable working as sunspot. Nevertheless, non-trivial behavioural uncertainty is not consistent with this idea because common knowledge that agents simultaneously believe in such unrelated variable is required for the emergence of a particular sunspot solution, for that any behavioral uncertainty is actually involved in the definition. Differently, here we want to link the possibility of correlated deviations from REE prescriptions to agents believe *possibly* non-trivial behavioral uncertainty is in play. To this aim expectation polarization hypothesis is introduced.

**Expectations polarization.** Let's start from the idea, consistent with statistical learning approach, that forming expectations is a costly activity at least form a cognitive point of view. It is unreasonable to assume that the most part of agents are expert in economics. It is more natural to sooner think they don't have a particular theory on how the economy works. Rather they rely on expectations of some more informed agent like a market leader, or a financial institution that has organizational skills and adequate resources to gather and rationally process information. Few institutional forecasters act as focal points for private sector expectations because economies of scale in the "production" of information are typically much stronger than in the production of any other good. The very small number of rating agencies in financial markets is an immediate example of this idea in real economy. In this sense institutional forecasters polarize public expectations.

Let' define formally the structural heterogeneity between an institutional forecasters and private sector. In sake of simplicity assume there are only two institutional forecasters forming expectations according to

$$E^i_{t-1}x_t \equiv \mathbf{E}[x_t|\Omega^i_{t-1}], \quad \forall i = 1, 2 \tag{5}$$

In words institutional forecasters maintain mathematical expectation of the generic process $x_t$ conditioned to available information up to time $t-1$. Assumption (5) is a formal specification of procedural rationality. It is natural to think professional forecasters are very few because information processing presents strong scale economy effects. We postpone the precise definition of $\Omega^i_{t-1}$ until the definition of their learning problem. Differently, the private sector have the following expectation function specification

$$E^z_{t-1}\pi_t = E^{i_z}_{t-1}\pi_t + v_{z,t-1}, \tag{6}$$

where $E^z_{t-1}(\cdot)$ is nothing else then an imitation correspondence and $i_z$ is the institutional forecaster noisily imitated by agent $z$ belonging to private sector agents set $Z$. Notice that the noise occurs since behavioral uncertainty assumption. If this working hypothesis is reasonable, agents expectations are polarized around few institutional forecasters' forecasts. The aggregate expectation in the economy is given by

$$E_{t-1}\pi_t = \int_{z\in Z} E^z_{t-1}\pi_t \; dz = \sum_{i=1,2} \lambda_i E^i_{t-1}\pi_t + \int_{z\in Z} v_z \; dz \ , \quad \sum_{i=1,2} \lambda_i = 1 \tag{7}$$

where $\lambda_i \in (0,1)$ represents the size of the public relying on agent $i$'s expectation. In the present work $\lambda$ is an exogenous parameter. The extent of agent $i$'s basin of audience, represented by $\lambda_i$, measures the average impact of agent $i$'s expectation on the aggregate expectations[3]. It would be very interesting to endogenize it with respect the performance of institutional forecasters performance. This route will be not undertaken in the present work. Nevertheless from here onward we focus on the case of two institutional forecasters polarizing evenly private sector ($\lambda_i = 1/2$). This assumption has a sense given observational error are equal and institutional forecasters both face the same problem, so that $\lambda_i = 1/2$ is a rest point of the replication dynamics driven by institutional forecasters' relative performance.

Equation (7) invalidates the negligibility of agents individual impact in the economy as assumed in general equilibrium perspective. In particular the impact of each institutional forecaster in the economy is equal and amounts half of the overall aggregate expectation effect. As long as expectations are strongly polarized strategic interaction motives arise among institutional forecasters in expectations formation. We are assuming each agent uses signal from an institutional forecaster to form expectations, therefore the noisy perception of institutional forecasters can provide information on a common factor embodied in agents' expectations identifying eventual correlated deviations.

**Information diffusion.** Figure 1 displays the information diffusion scheme entailed by assumptions above. Two institutional forecasters (red points) affect aggregate expectation calculated over an ocean of agents according to their respective basin of audience supposed to be equal. The aggregate expectation yields an actual inflation level as implied by (2). Moreover they have noisy perceptions of the other institutional forecaster's simultaneous expectations. Arrows show flows of information. The two institutional forecasters analyse available data with statistical tools and, on the basis of their estimates, form expectation on future actual inflation.

Three are the key coefficients of the model, as remarked in the picture: $\beta$ is the feedback of aggregate expectation of current inflation on actual inflation, $\rho_v$ is the correlation coefficient between institutional forecasters' observational errors, and finally $\gamma$ denotes the covariance between institutional forecasters' expectations and the individual observational error committed by the private sector in the noisy imitation. The latter measures the non-neutrality of information transmission channel and will be conveniently defined later. Inflation dynamics is affected by learning determinants, that is, how institutional forecasters' expectations evolve in time, and communications determinants, that is what happens to information in flowing from institutional forecasters to private sector. We will see soon in next section how those three parameters are

[3] The ones filling uncomfortable with mere imitation implied by (6) can imagine more sophisticated expectation-takers agents weighting all expectations-makers' expectations. From this perspective $\lambda_i$ would be interpreted as the public average consideration of agent $i$ in expectations formation. This would not change the substance and mathematics of following arguments.

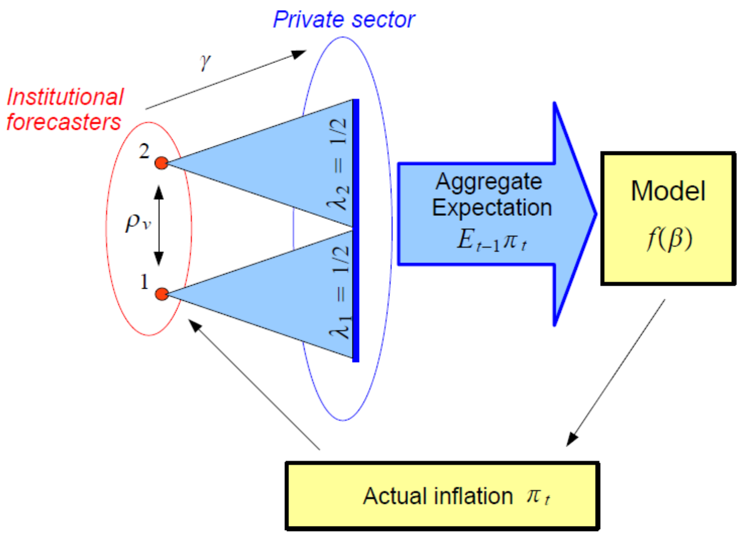


Figure 2: Information diffusion in the economy. Institutional forecasters 1 and 2 analyse data as they become available in time and produce statistically optimal forecasts. Institutional forecasters' expectations polarize evenly private sector expectations. The latter determine, jointly with other exogenous determinants, the actual inflation.

enough to grasp basic phenomena arising from the interaction of learning (about fundamentals and rationality of others) and institutional communication.

The emergence of the unique REE depends on the game played by institutional forecasters. Given the power of each institutional forecaster to displace actual inflation away from fundamentals, holding rational expectation is a best expectation if and only if each institutional forecasters believe the other one hold rational expectations. In order to satisfy this requirement institutional forecasters have a double task: learning about fundamentals and learning whether or not the other institutional forecaster, and so a non trivial part of agents, has rational expectations.

# 3 From perceived to actual law of motion

## 3.1 Learning determinants

Institutional forecasters learn about a long-run exogenous component of inflation process driven by fundamentals, and about an eventual endogenous idiosyncratic component due to possibly idiosyncratic deviations of the other institutional forecaster's expectation from the rational one.

**Learning about fundamentals: the exogenous long-run component.** The fundamental inflation rate is the long-run component of inflation, denoted by $\overline{\pi}_t$, determined by truly exogenous components. This is the only process compatible with long run equilibrium of agents' forecasts, that is, with rational expectations. Institutional forecasters learn about the fundamental inflation rate regressing a constant and the relevant exogenous variables affecting the economy on actual inflation, namely, in our case, respectively $\mathbf{z}'_{t-1} \equiv [1, \omega_{t-1}]$, on inflation $\pi_t$. As standard in adaptive learning literature we assume they hold a correct perceived law of motions encompassing REE form

$$E^i_{t-1}\overline{\pi}_t = \mathbf{a}'_{i,t-1}\mathbf{z}_{t-1}$$

where $\mathbf{a}'_{i,t-1} \equiv [a^c_{i,t-1}\ a^{\omega}_{i,t-1}]$ are estimated coefficients. Specifically we assume $\mathbf{a}'_{i,t-1}$ is updated recursively in time according to the following constant stochastic gradient (CSG) rule

$$\mathbf{a}_{i,t-1} = \mathbf{a}_{i,t-2} + g_f\ \mathbf{z}_{i,t-1}\left(\pi_{t-1} - \mathbf{a}'_{i,t-1}\mathbf{z}_{t-1}\right), \tag{8}$$

where $g$ is a constant gain smaller than one. Asymptotically the two estimates coincides independently on possibly different initial priors. Therefore we can consider the two as having the same estimate for fundamental inflation, that we label $\overline{\pi}^e_t$, with approximation vanishing very soon. Algorithm (8) is similar to the recursive version of OLS where the estimated correlation matrix is settled equal to the identity matrix and the gain coefficient is fix[4]. CSG converges to an ergodic distribution centred on the rest point of the T-map in the same cases recursive OLS would punctually converge. CSG, as any constant gain learning rule, exhibits permanent learning since a bigger weight is given to more recent data. This makes these class of algorithms to be particularly suitable for learning structural changes. Recursive OLS on the contrary converges at the cost of a huge stickiness of the dynamics for $t$ big enough. Moreover CSG algorithm are also derived as optimal solution to a forecast errors variance minimization problem, given agents are "sensitive" to risk in a particular form. For details see Evans, Honkapohja and Williams (2005).

**Learning about others' rationality: the endogenous idiosyncratic component.** Even in case institutional forecasters correctly estimate fundamental inflation, actual inflation differs from the fundamental one at least for the exogenous stochastic noise $\nu_t$. Nevertheless, because non trivial behavioral uncertainty is in place, institutional forecasters cannot exclude such stochastic deviations are due to idiosyncratic departure of aggregate expectation form the rational one. In particular, both institutional forecasters have to understand whether or not deviations from the REE are due to deviations of the other institutional forecasters' expectation from the rational expectation. In other words, they have to understand if the signal about others' expectations is informative about such departures. To this aim, they estimate the optimal weight to give to noisy observations they have in order to refine their forecasts on actual idiosyncratic

[4] CSG is obtained form recursive constant gain OLS formula

$$\begin{aligned} b_{t-1} &= b_{t-2} + \overline{g}R^{-1}_{t-1}z_{t-2}(\pi_t - E^1_{t-2}\pi_t) \\ R_{t-1} &= R_{t-2} + \overline{g}\left(z^2_{t-2} - R_{t-2}\right) \\ z_{t-2} &= E^2_{t-2}\pi_t + v_{1,t-2} - \overline{\pi}^e_{t-1} \end{aligned}$$

fixing $R_{t-1} = \mathbf{1}$. Therefore, in order to obtain adjustments comparable with constant gain OLS we have to rescale the gain such that $g = \overline{g}/var(z_t)$ given $\lim_{t\to\infty} R_{t-1} = var(z_t)$.

inflation deviations from the fundamental. In this perspective they extract the signal of others' rationality in real time whenever they "recognize" as useless the noisy information about others' expectations. They follow the rule

$$E^1_{t-1}\left(\pi_t - \overline{\pi}^e_t\right) \;=\; b_{t-1}\;\phi^1_{t-1}, \tag{9a}$$

$$E^2_{t-1}\left(\pi_t - \overline{\pi}^e_t\right) \;=\; c_{t-1}\;\phi^2_{t-1}, \tag{9b}$$

where agents have to set $b_{t-1}$ and $c_{t-1}$ in order to minimize their forecast error variance according to the same constant gain scheme agents use for forecasting the fundamental inflation

$$\begin{aligned} b_{t-1} &= b_{t-2} + g_d\;\phi^1_{t-1}\;(\pi_{t-1} - E^1_{t-2}\pi_{t-1}), \\ c_{t-1} &= c_{t-2} + g_d\;\phi^2_{t-1}\;(\pi_{t-1} - E^2_{t-2}\pi_{t-1}), \end{aligned}$$

where $g_d \leq g_f$ are the updating gains, $(\pi_t - E^i_{t-2}\pi_t)$ is the forecast error and $\phi^i_{t-1} = [E^j_{t-1}\pi_t + v_{i,t-1} - \overline{\pi}^e_t]'$ is the noisy observed displacements of others' expectations from the estimated fundamental one. In words, they regress the nosily observed displacement of others' expectation from the estimated fundamental on displacements of the actual inflation from the estimated fundamental one.

If estimates $(b_{t-1}, c_{t-1})$ asymptotically converge to zero, dependence from noisy observation of simultaneous expectations is rejected. In such a case agents will forecast the fundamental value, so that aggregate expectation will be a rational expectations. In other words, if all institutional forecasters are rational they would not need to condition their expectations on noisy observations of the other one's simultaneous expectations. But if this is not the case considering noisy observations do improve the accuracy of forecasts. Learning is valuable exactly because this form of behavioral uncertainty is introduced. In this case, CSG has the advantages of showing convergence to equilibria and, at the same time, testing the possibility of endogenous and unpredictable shifts from the rational expectations equilibrium to a BSE.

Institutional forecasters have incentive to learn also about others' rationality, because, differently from the adaptive learning standard setting, they do recognize the self-referential nature of the problem. They are not bounded rational as generally assumed. This possibility is a novelty in the approach introduced by Gaballo (2009) that here we are going to implement in a fully fledged monetary model, reconciling this scheme to standard use of adaptive learning.

### 3.2 Communication determinants

**Non-neutrality of the transmission channel**. Before to analyse how institutional forecasters expectations impact on the actual law of motion we have to specify the process $v_z$ in order to evaluate the effect of information transmission from institutional forecasters to private sector formally given by $\int_{z\in Z} v_z\; dz$ present in (7). In particular it is convenient to express $v_z$ in the following way

$$v_z = \gamma(E^{i_z}_{t-1}\pi_t - \overline{\pi}_t) + (1-\gamma_z)\,\epsilon_{z,t} \tag{10a}$$

where $\epsilon_{z,t}$ is a i.i.d. shock distributed according $\Upsilon\left(0,\left(1-\gamma^2\mathbf{E}(E^{i_z}_{t-1}\pi_t-\overline{\pi}_t)^2\right)/\left(1-\gamma\right)^2\right)$ so that $\mathbf{E}\left(v_z^2\right)=\delta$. The specific forms of observational errors as maintained by (10) don't add nothing of substantial in the general framework. The coefficient $\gamma$ controls for the covariance between this observational error, and the estimated distance of the actual inflation from the fundamental one. Notice the latter is a proxy for the amount of behavioural uncertainty in the economy. In other words, this specification takes account of the idea that the public receives a biased information whose idiosyncratic component is possibly further amplified or dumped in the transmission. Finally notice that

$$\int_{z\in Z} v_z\ dz=\frac{\gamma}{2}\left(\sum_{i=1,2}\left(E^i_{t-1}\pi_t-\overline{\pi}^e_t\right)\right), \tag{11}$$

so that the aggregate expectation of $E_{t-1}\pi_t$, is now equal to

$$E_{t-1}\pi_t=\overline{\pi}^e_t+\frac{(1+\gamma)}{2}\sum_{i=1,2}\left(E^i_{t-1}\pi_t-\overline{\pi}^e_t\right). \tag{12}$$

### 3.3 The actual law of motion

From (9) it is simple to check PLMs can be expressed as linear function of the fundamental price and observational errors. Formally we have

$$E^1_{t-1}\pi_t = \overline{\pi}^e_t+\frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1}+\frac{b_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1} \tag{13a}$$

$$E^2_{t-1}\pi_t = \overline{\pi}^e_t+\frac{b_{t-1}c_{t-1}}{1-b_{t-1}c_{t-1}}v_{1,t-1}+\frac{c_{t-1}}{1-b_{t-1}c_{t-1}}v_{2,t-1} \tag{13b}$$

provided $bc\neq 1$. Processes (13) cannot be inferred by agents since they cannot disentangle observational errors. According to (2), (12) and (13), specifications (13) makes the actual law of motion to move according to the following process

$$\pi_t=\alpha\omega_{t-1}+\beta\overline{\pi}^e_t+\beta^*\left(\frac{b_{t-1}\left(1+c_{t-1}\right)}{1-b_{t-1}c_{t-1}}v_{1,t-1}+\frac{c_{t-1}(b_{t-1}+1))}{1-b_{t-1}c_{t-1}}v_{2,t-1}\right)+\eta_t \tag{14}$$

where $\beta^*\equiv\beta(1+\gamma)$. Notice that $\pi_t=\overline{\pi}^e_t$ if: *i*) agents are not uncertain about others' behavior, that is $v_{1,t-1}=0$ and $v_{2,t-1}=0$, or *ii*) agents hold the rational expectation, that is, $b^1_{t-1}=0$ and $c_{t-1}=0$, or *iii*) expectations have a zero impact on the actual course given $\beta^*=0$. Otherwise inflation will exhibit endogenous excess volatility around the estimated inflation due by the stochastic term multiplied by $\beta^*$. Gaballo (2009) proved that such excess volatility learneable equilibria exists. Here we want to defend, in principle, the economic relevance of such high volatility regimes, and more importantly, we want to show how constant gain learning can generate endogenous and unpredictable switches from REE to BSE and viceversa.

# 4 Equilibria and Learneability

## 4.1 Equilibria

Equilibria are such that institutional forecasters' forecast errors are orthogonal to available information, namely to exogenous variables time series and noisy perceptions of others' expectations. Formally they have to solve the following system

$$\begin{aligned}
\mathbf{E}[\mathbf{z}_{t-1}\left(\pi_t - T_{\mathbf{a}}\mathbf{z}_{t-1}\right)] &= 0 && (15)\\
\mathbf{E}[\phi^1_{t-1}\left(\pi_t - \overline{\pi}^e_t - T_b\phi^1_{t-1}\right)] &= 0 && (16)\\
\mathbf{E}[\phi^2_{t-1}\left(\pi_t - \overline{\pi}^e_t - T_c\phi^2_{t-1}\right)] &= 0 && (17)
\end{aligned}$$

where $T.$ map is a function giving the coefficients of the linear forecast of $\pi_t$ yielding the smaller mean square error variance conditioned on the available information set. For a technical reference on projections and convergence properties of adaptive learning algorithms used in what follows, see Marcet and Sargent (1989), Evans and Honkapohja (2001).

**Proposition 1** *T-map takes the form*

$$\begin{aligned}
T_{\mathbf{a}} &= \alpha\omega_{t-1} + \beta\left(\mathbf{1}'\mathbf{a}\right)\\
T_b &= \frac{\beta^*}{2}\left(\frac{b\left(1+c\right)\left(1+c\rho_v\right) + c(1+b)\left(c+\rho_v\right)}{1+c^2\left(1+2\rho_v\right)}\right)\\
T_c &= \frac{\beta^*}{2}\left(\frac{b\left(1+c\right)\left(b+\rho_v\right) + c(1+b)\left(1+b\rho_v\right)}{1+b^2\left(1+2\rho_v\right)}\right)
\end{aligned}$$

**Proof.** Keep in mind that observational errors $v^i_{t-1}$ have zero mean and they are uncorrelated with exogenous variable $\omega_{t-1}$. Spelling out conditions for $T_{\mathbf{a}}$ and $T_b$ ($T_c$ is mirror like),we have respectively

$$T_{\mathbf{a}} : [0,\alpha]E[\mathbf{z}^2_{t-1}] + \beta\left(\mathbf{1}'\mathbf{a}\right)E[\mathbf{z}^2_{t-1}] - T_{\mathbf{a}}E[\mathbf{z}^2_{t-1}] = 0$$

and

$$\begin{aligned}
T_b : \frac{\beta^*}{2}&\left(\frac{b\left(1+c\right)}{\left(1-bc\right)^2}\delta + \frac{c^2(b+1)}{\left(1-bc\right)^2}\delta + \frac{c(b+1)+cb\left(1+c\right)}{(1-bc)^2}\delta\rho_v\right) +\\
&- T_b\left(\frac{1}{\left(1-bc\right)^2}\delta + \frac{c^2}{\left(1-bc\right)^2}\delta + 2\frac{c}{\left(1-bc\right)^2}\delta\rho_v\right) = 0
\end{aligned}$$

or after same simple rearranging

$$\begin{aligned}
T_b : \frac{\beta^*}{2}&\left(\frac{b\left(1+c\right)}{\left(1-bc\right)^2}\left(1+c\rho_v\right)\delta + \frac{c(b+1)}{\left(1-bc\right)^2}\left(c+\rho_v\right)\delta\right) +\\
&- T_b\left(\frac{1}{\left(1-bc\right)^2}\delta + \frac{c}{\left(1-bc\right)^2}\left(c+2\rho_v\right)\delta\right) = 0.
\end{aligned}$$

Finally the projected $T$map for $\mathbf{a}$, $b$ and $c$ is given by

$$\begin{aligned}
T_{\mathbf{a}} &= \alpha + \beta\left(\mathbf{1}'\mathbf{a}\right) \\
T_b &= \frac{\beta^*}{2}\left(\frac{b\,(1+c)\,(1+c\rho_v)+c(b+1)\,(c+\rho_v)}{1+c^2+2c\rho_v}\right), \\
T_c &= \beta^*\left(\frac{b\,(1+c)\,(b+\rho_v)+c(b+1)\,(1+b\rho_v)}{1+b^2+2b\rho_v}\right).
\end{aligned}$$

■

Notice T-map depends on error variances ratio $\varepsilon_1/\varepsilon_2$ and not at all on the extent of them. Moreover if $\varepsilon_1=\varepsilon_2$ errors variances simply disappear from equations. Fix points of the T-map give the optimal $(b,c)$ such that professional forecasters don't commit systematic error given available information.

**Definition 2** *Equilibria obtain as fix points of the $T$ map for $T_{\mathbf{a}}(\widehat{\mathbf{a}})=\alpha$, $T_b\left(\widehat{b},\widehat{c}\right)=\widehat{b}$ and $T_c\left(\widehat{b},\widehat{c}\right)=\widehat{c}$.*

Now it is possible to state the following.

**Proposition 3** *Equilibria of the system are:*
*i) a REE $(\widehat{\mathbf{a}}',\widehat{b},\widehat{c})=(0,\frac{\alpha}{1-\beta},0,0)$,*
*ii) an high BSE $(\widehat{\mathbf{a}}',\widehat{b},\widehat{c})=(0,\frac{\alpha}{1-\beta},\frac{\beta^*-(2-\beta^*)\rho_v+2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)},\frac{\beta^*-(2-\beta^*)\rho_v+2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)})$,*
*iii) a low BSE $(\widehat{\mathbf{a}}',\widehat{b},\widehat{c})=(0,\frac{\alpha}{1-\beta},\frac{\beta^*-(2-\beta^*)\rho_v-2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)},\frac{\beta^*-(2-\beta^*)\rho_v-2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)})$.*
*The unique REE arises for any feasible calibration, whereas both BSE exist only for $\beta^*\geq 1$.*

**Proof.** Equilibria are given by the system:

$$\widehat{\mathbf{a}}' = \left[0,\ \frac{\alpha}{1-\beta}\right] \tag{18a}$$

$$\widehat{b} = \frac{(\beta^*/2)\,\widehat{c}(\widehat{c}+\rho_v)}{(1-(\beta^*/2)\,(1-\rho_v)\,)\widehat{c}^2+((2-\beta^*)\rho_v-\beta^*/2)\widehat{c}+(1-(\beta^*/2)\,)} \tag{18b}$$

$$\widehat{c} = \frac{(\beta^*/2)\,\widehat{b}(\widehat{b}+\rho_v)}{(1-(\beta^*/2)\,(1-\rho_v)\,)\widehat{b}^2+((2-\beta^*)\rho_v-\beta^*/2)\widehat{b}+(1-(\beta^*/2)\,)} \tag{18c}$$

assuming $bc\neq 1$. It is easily proved by substitution that the fundamental rational expectation solution $(0,0)$ it is always a rest point of the T-map. Other non fundamental $b$ and $c$ equilibria values are in correspondence of $\widehat{b}=\widehat{c}$ and result as solutions to

$$c\left(c^2\left((1-\beta^*/2)-(\beta^*/2)\,\rho_v\right)-\left(\beta^*-(2-\beta^*)\,\rho_v\right)c+(1-\beta^*/2)-(\beta^*/2)\,\rho_v\right)=0 \tag{19}$$

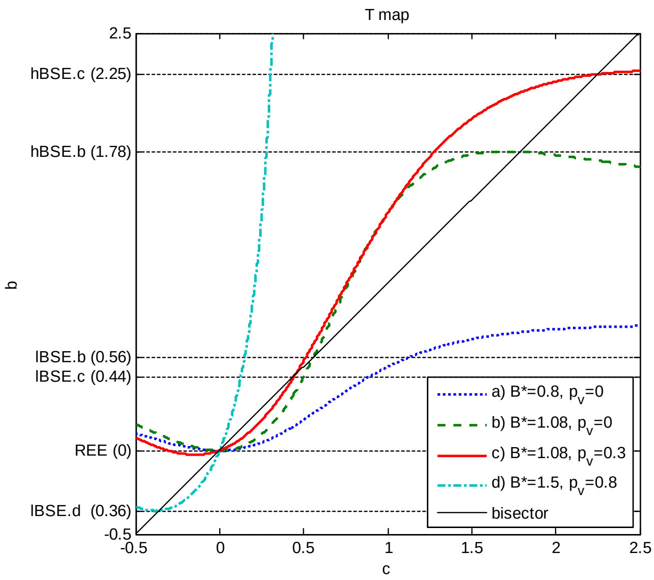


Figure 3: Tmap representation for different calibration. Equilibria are at intersection with Tmap with the bisector. For values of $\beta^*$ bigger than one two BSEs arise besides REE.

featuring respectively the high BSE values $(b_+, c_+)$ and the low BSE values $(b_-, c_-)$ where

$$c_+ = b_+ = \frac{\beta^* - (2-\beta^*)\rho_v + 2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)} \tag{20}$$

$$c_- = b_- = \frac{\beta^* - (2-\beta^*)\rho_v - 2\sqrt{(\beta^*-1)(1-\rho_v^2)}}{2-\beta^*(1+\rho_v)} \tag{21}$$

exist whenever $\beta^* \geq 1$. ■

The figure below plots $T_b$ for four different calibrations. Given the symmetric nature of the problem we are analyzing, BSEs are at the intersection of $T_b$ with bisector. Line $a$) is obtained for $\beta^* = 0.8$ and $\rho_v = 0$. In such a case the unique intersection is at the REE values $\widehat{b} = \widehat{c} = 0$. As $\beta^*$ goes up to one (line $b$)), two BSE emerge such that not trivial values of $\widehat{b}$ and $\widehat{c}$ exist such that forecast errors are uncorrelated with available information (see Gaballo 2009 for further details). Ceteris paribus increasing values of $\rho_v$ (line $c$)) makes the BSE with smaller values (henceforth low BSE) closer to REE values and the high one being further away. Finally, extreme calibration as the one showed by line $d$) yields negative BSEs; the low one is shown in the picture.

The key feature making REE and BSE to coexist is the non-linearity of the T-map. Notice that BSE are kind of limited-informed rational expectations (Sargent 1991) in that agents cannot observe separately all the stochastic components of the actual law of motion. However, differently from the standard setting proposed there, here the subjective information set results in a coarser set, rather than a subset, of all the relevant exogenous variables. This occurs because observational errors are correlated by non avoidable non linear constraints resulting from the system (9). This technical feature is at the basis of non linearity of the Tmap and the emergence of BSE.

It is worthwhile to discuss economic insight underlying the existence of BSE in reference to classical sunspots equilibria. Both in BSEs and classical sunspot equilibria there are additional variables uncorrelated with exogenous variables of the model. They differ for the kind of belief required to self-sustain them. In classical sunspot equilibria is required agents to hold common knowledge that a certain observable extra variable affects the equilibrium. Differently, BSE require all agents must believe[5] that displacements of the actual output from the fundamental one depends on others' deviation from rational expectations. BSE take shape as a coordination failure. After a certain threshold, for agent 1 it is a best action to condition her own expectations to noisy observations of agent 2's expectation, given agent 2 is doing the mirror-like action. Specifically, an agent conditioning her own expectation on noisy signal of the other agent's expectation results to be as if it was "irrational exuberant"; in turn, an agent testing irrational exuberance of the other agent has incentive to condition her own expectation to the noisy observation of the other agent's expectations. When a BSE is achieved, observational errors variance transmits persistently to actual output course through aggregate expectation. This mechanism could give new insights on how behavioral uncertainty can trigger excess volatility phenomena. The exercise at hand shows that this issue is relevant *even if* agents know everything about the exogenous determinants of the economy.

## 4.2 Learneability

This section explores leaneability of REE and the possibility of adaptive learners being stuck in a BSE, that is whether or not BSE are learneable . The concept of learneability refers to the nature, stable or unstable of the learning dynamics under a recursive least square algorithm around the equilibria computed above.

**Definition 4** *An equilibrium* $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *is locally learnable under recursive least square (RLS) algorithm if and only if there exist some neighborhood* $\Im(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *of* $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *such that for each initial condition* $(\mathbf{a}_0,b_0,c_0)\in\Im(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ *the estimates converge almost surely to the equilibrium, that is* $\lim_{t\to\infty}(\mathbf{a}_{t-1},b_{t-1},c_{t-1})\overset{a.s.}{=}\left(\widehat{\mathbf{a}},\widehat{b},\widehat{c}\right)$.

To check learneability one need to investigate the Jacobian of the Tmap. If the matrix of all partial derivative of Tmap in the equilibrium has all eigenvalues in the unit root, we can say the equilibrium to be stable under learning (Marcet and Sargent 1989, Evans Honkapohja 2001). The Jacobian for T-map takes the form

$$JT(\mathbf{a},b,c)=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \frac{dT_b(b,c)}{db} & \frac{dT_b(b,c)}{dc}\\ 0 & 0 & \frac{dT_b(b,c)}{dc} & \frac{dT_c(b,c)}{dc}\end{pmatrix}$$

[5] Notice that it is not required that they believe that others believe.

where

$$\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}b} = \frac{\beta^*}{2}\frac{(1+c)(1+c\rho_v)+c(c+\rho_v)}{1+c^2+2c\rho_v}, \tag{22a}$$

$$\frac{\mathrm{d}T_b(b,c)}{\mathrm{d}c} = \frac{\beta^*}{2}\frac{b(1+c\rho_v)+b\rho_v\,(1/2)\,(1+c)+(c+\rho_v)(1+b)}{1+c^2+2c\rho_v}+ \\ -\frac{2(c+\rho_v)(b(1+c\rho_v)(1+c)+c(c+\rho_v)(1+b))}{1+c^2+2c\rho_v}, \tag{22b}$$

$$\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}b} = \frac{\beta^*}{2}\frac{c(1+b\rho_v)+c\rho_v\,(1/2)\,(1+b)+(b+\rho_v)(1+c)}{1+b^2+2b\rho_v}+ \\ -\frac{2(b+\rho_v)(c(1+b\rho_v)(1+b)+b(b+\rho_v)(1+c))}{1+b^2+2b\rho_v}, \tag{22c}$$

$$\frac{\mathrm{d}T_c(b,c)}{\mathrm{d}c} = \frac{\beta^*}{2}\frac{(1+b)(1+b\rho_v)+b(b+\rho_v)}{1+b^2+2b\rho_v}. \tag{22d}$$

Now let's analyse learneability of equilibria. To this aim we have to investigate the sign of eigenvalues of the matrix $K \equiv JT - I$ (where $I$ is the identity matrix) in the equilibrium values $\widehat{\mathbf{a}}$ and $\widehat{c} = \widehat{b}$

$$K_{(\widehat{\mathbf{a}},\widehat{b},\widehat{c})} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta - 1 & 0 & 0 \\ 0 & 0 & \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} - 1 & \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} \\ 0 & 0 & \left[\frac{dT_c(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} & \left[\frac{dT_c(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} - 1 \end{pmatrix}, \tag{23}$$

with

$$\begin{aligned} \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} - 1 &= \frac{((\beta^*/2)\,(1+\rho_v)-1)\,\widehat{b}^2 + ((\beta^*/2)(1+2\rho_v)-2\rho_v))\,\widehat{b} + (\beta^*/2) - 1}{1+\widehat{b}^2+2\widehat{b}\rho_v} \\ \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} &= (\beta^*/2)\,\frac{(2\rho_v^2-1)\widehat{b}^3 + 3\rho_v^2\widehat{b} + 3\widehat{b} + \rho_v}{\left(1+\widehat{b}^2+2\widehat{b}\rho_v\right)^2} \\ \left[\frac{dT_c(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} &= \left[\frac{dT_b(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} \quad \text{and} \quad \left[\frac{dT_c(b,c)}{db}\right]_{(\widehat{b},\widehat{c})} = \left[\frac{dT_b(b,c)}{dc}\right]_{(\widehat{b},\widehat{c})} \end{aligned}$$

A certain equilibrium $(\widehat{\mathbf{a}},\widehat{b},\widehat{c})$ is learnable if and only if the matrix $K_{(\widehat{\mathbf{a}},\widehat{b},\widehat{c})}$ has all negative eigenvalues. The picture below shows numerical analysis for the whole parameter range[6] spanned by $\beta^*$ and $\rho_v$. Keep in mind that a necessary conditions is always $\beta < 1$.

[6] From quite immediate application of a proposition proved in Gaballo (2009), REE solution $(\widehat{\mathbf{a}},\widehat{b},\widehat{c}) = (0, \frac{\alpha}{1-\beta}, 0, 0)$ is learneable whenever

$$\begin{aligned} \beta &< 1, & \quad (24) \\ \beta^* &\leq \frac{2}{1+\rho_v} \quad \text{with } \rho_v \geq 0, & \quad (25) \\ \beta^* &\leq \frac{2}{1-\rho_v} \quad \text{with } \rho_v < 0. & \quad (26) \end{aligned}$$

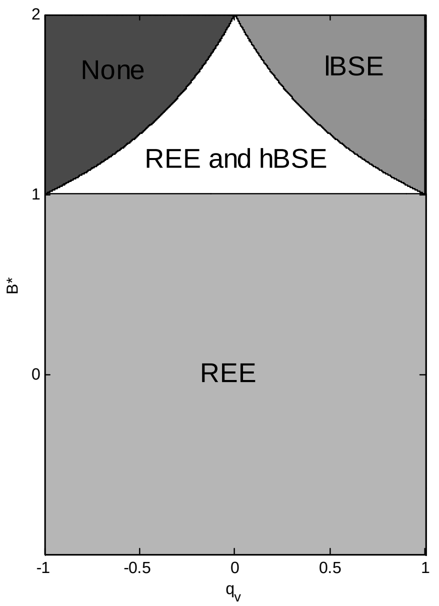


Figure 4: Numerical learneability analysis in the whole parameter space. The space is partitioned in four regions exhibiting different learneability properties. In the white one REE and the high BSE are both learneable and learning dynamics select among them. In the light grey only REE is learneable whereas in the dark grey only the low BSE is learneable. In the black area none learneable equilibria are present.

As is evident from inspection of the figure REE is the only learneable equilibrium in the region $\beta^* < 1$. In the white area a learneable high BSE (hBSE) arises besides a REE. This area is the most interesting in that it partially includes most realistic calibration values for the Lucas-type monetary model. Notice that whenever a hBSE exists it is not the unique learneable equilibrium. For such values the learning mechanism selects between REE and hBSE. How this happens will be explained in detail later, when we will run numerical simulation of the dynamic system. REE and hBSE are both learneable for lower values of $\beta^*$ as $\rho_v$ increases in modulus. Specifically, as $\rho_v$ approach unity for sufficiently high value of $\beta^*$ the low BSE (lBSE) becomes learneable and both REE and hBSE are never more. This is a standard property of non linear dynamics: given the system has three equilibria, either the most distant two are dynamically stable or only the one in the middle is dynamically stable (refer to figure 3). On the other hand as $\rho_v$ decreases for sufficiently high value of $\beta^*$ the system presents no learneable equilibria.

### 4.3 Excess volatility

Equilibria with $\left(\widehat{b}, \widehat{c}\right) \neq (0, 0)$ were tagged Behavioral Sunspot Equilibria in Gaballo (2009). The word sunspot refers to the presence of an extra stochastic component in agents' expectation function, namely observational errors. BSE present, as any sunspot solution, a volatility higher

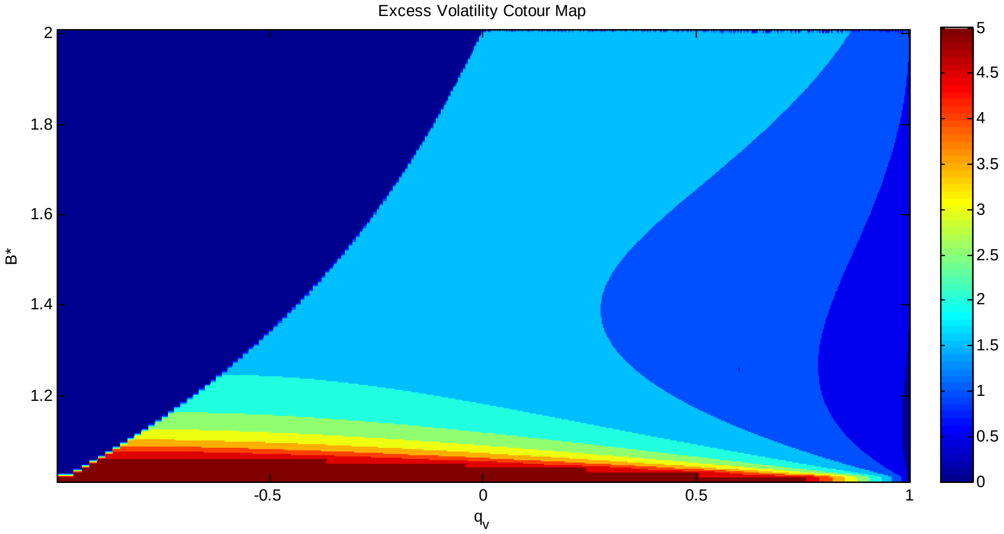


Figure 5: Numerical analysis of excess volatility. The picture shows the size of excess volatility obtained for values $\beta^* \in (1, 2)$ for which learneable BSEs arise. The unit of measure of the scale is the variance of observational errors. ( "5" stays for "5 and more" ).

than the fundamental solution. The extent of theoretical excess variance is measured by

$$\left(\frac{\beta^*}{2}\right)^2 \left(\frac{b^2 (1+c)^2}{(1-bc)^2} + \frac{c^2(b+1)^2}{(1-bc)^2} + 2\frac{bc(b+1)(1+c)}{(1-bc)^2}\rho_v\right)\delta \tag{27}$$

and it is increasing in $\beta$, $\delta$ and $\rho_v$. In equilibrium $\left(\widehat{b} = \widehat{c}\right)$ (27) is measured by

$$2\left(\frac{\beta^* \widehat{b}}{2\left(1-\widehat{b}\right)}\right)^2 (\varrho + \rho_v),$$

and it is decreasing in $\widehat{b}$ for $\widehat{b} > 1$.

The picture below plots excess variance yield by learneable BSE in terms of observational error variance for values $\beta^* \in (1, 2)$ ("5" stays for "5 and more"). For values close to unity excess variance is really high but it decreases very soon. The most part of the relevant region exhibits an excess volatility between one and four times the variance of observational errors. In the region on which REE and the high BSE are both learneable we can have different volatility regimes (an high volatility one being in correspondence of the high BSE) depending on the equilibrium selected by the learning algorithm. Next section we finally explain and show how unpredictable and endogenous switching of volatility regimes can be triggered by constant gain algorithm.

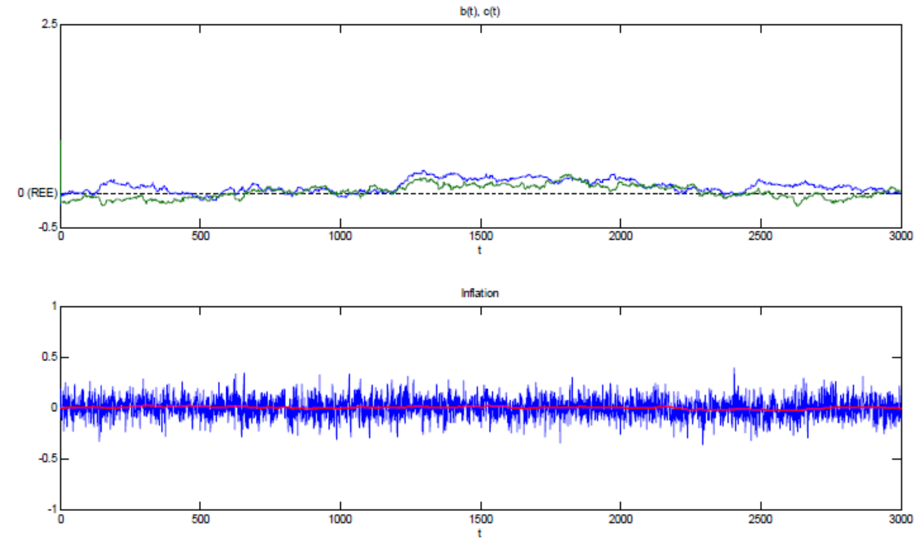


Figure 6: Benchmark case. Convergence to REE ($\beta = 0.8$, $\gamma = 0$, $\rho_v = 0$). Line a). in figure 3.

## 5 Constant gain learning simulation

Simulations proposed in this section have the goal to provide examples of endogenous and unpredictable changes in volatility regimes. We chose calibrations such that analytical results can be contrasted with experiments. The following parameters are set equal in all simulations: $\beta = 0.8$, $\delta = \varphi_2$, $\delta = 0.1$. The exogenous shocks are all Gaussian white noises with unitary variance. In all figures the following conventions hold. In the upper box is displayed the dynamics of the two coefficients, $b_t$ and $c_t$. The lower box shows the corresponding dynamics of both actual inflation $\pi_t$ (the blue line) and agents' estimated fundamental inflation (red flatter line). The first four figures are generated with the same series of errors and with initial conditions closely around REE value. Figure 6 represents the benchmark case, that is, convergence in distribution to REE values for $\rho_v = \gamma = 0$. The gain is settled $g_d = g_f = \delta^{-1}/110$. The factor $\delta^{-1}$ has been included in the gain so that the adjustments of both $b_t$ and $c_t$ dynamics are substantially equal to the one obtained with constant gain OLS around REE values for $\overline{g} = 110$.. Notice how constant gain learning can generate continuous even if small displacements away from REE values. Nevertheless such diplacements are temporary escapes and do not substantially affect the variance of actual inflation process. In the second box one can appreciate also how the estimate of fundamental inflation evolves in time (flatter red line).

In figure 6 the calibration of figure 1 is modified only in that $\gamma = 0.28$ so that $\beta^* = 1.08$. For such values one learneable high BSE arise for $b = c = 1.78$. Up to 1300 periods the dynamics is roughly the same, but how estimates approach low BSE values, the dynamics changes dramatically. In particular as estimates overcome low BSE values (around 2000 periods) the dynamics enters in the basin of attraction of the high BSE making estimates to converge in distribution to

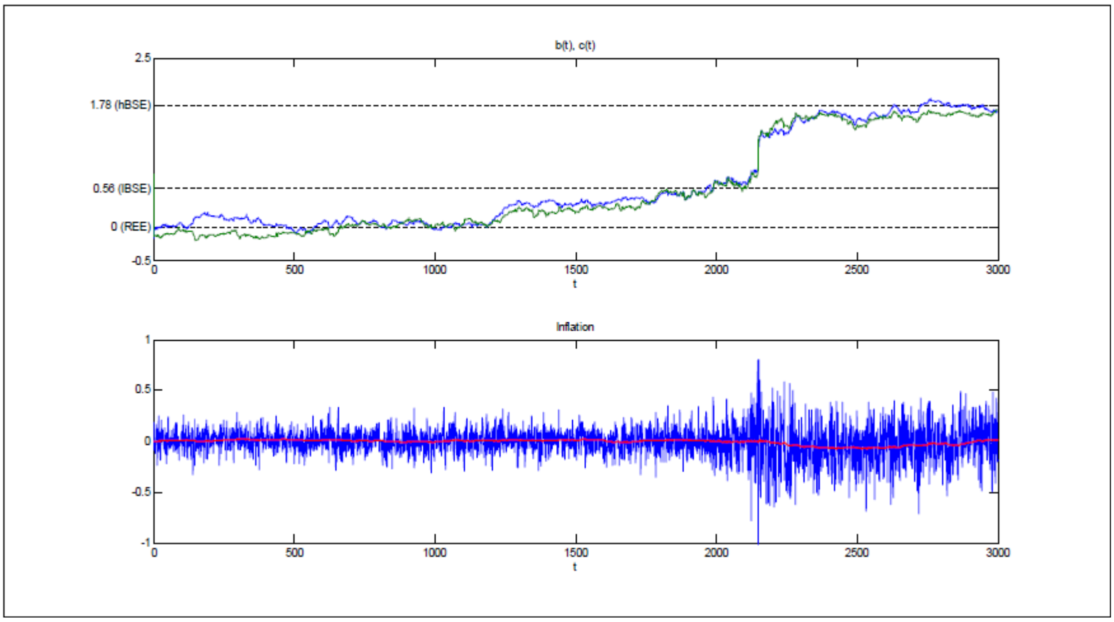


Figure 7: From REE to hBSE ($\beta = 0.8$, $\gamma = 0.28$, $\rho_v = 0$). Line b). in figure 3.

it. This endogenous structural change affect in a persistent and substantial way actual inflation variance. The resulting excess variance is about three times REE variance. Notice how excess volatility generated by high BSE affects volatility of the estimated fundamental inflation too, contributing to the overall variance of actual inflation. This effect is more evident in next and last pictures.

Figure 7 is generated with same setting introducing a small correlation between observational errors $\rho_v = 0.3$. The effect of this type of correlation is in a earlier jump to the correspondent learneable high BSE. This is not surprising since for increasing positive values of $\rho_v$ the corresponding high BSE values increase and low BSE ones decrease. This means that high BSE basin of attraction enhances and REE basin shrinks, so that jump from REE to high BSE is more likely to happen. One may want to appreciate such feature contrasting line b) and c) in figure 3 where distances between equilibria on the bisector line are indicative of the size of basins of attraction.

Figure 8 is to show an example of convergence to the low BSE. This occurs for quite extreme and careful calibration in that low BSE basin of attraction is quite narrow given the closeness of low BSE values to REE ones. The one displayed is obtained for $\gamma = 0.7$ and $\rho_v = 0.8$. As evident the contribute to overall actual inflation variance is almost negligible. Finally last picture shows how with appropriate calibration is it possible to obtain a series of endogenous and unpredictable switches from REE to high BSE and viceversa. Several features contribute to the aim. Firstly correlation coefficient are $\gamma = 0.21$ (that makes $\beta^* = 1.01$ very near unity) and $\rho_v = 0.4$. For such values low BSE values are about half of high BSE ones, that in turn result to be quite small. Therefore REE and high BSE basins of attraction have almost the same extent. Moreover we chose a bigger gain, namely $g_d = g_f = \varrho^{-1}/48$ in order to make

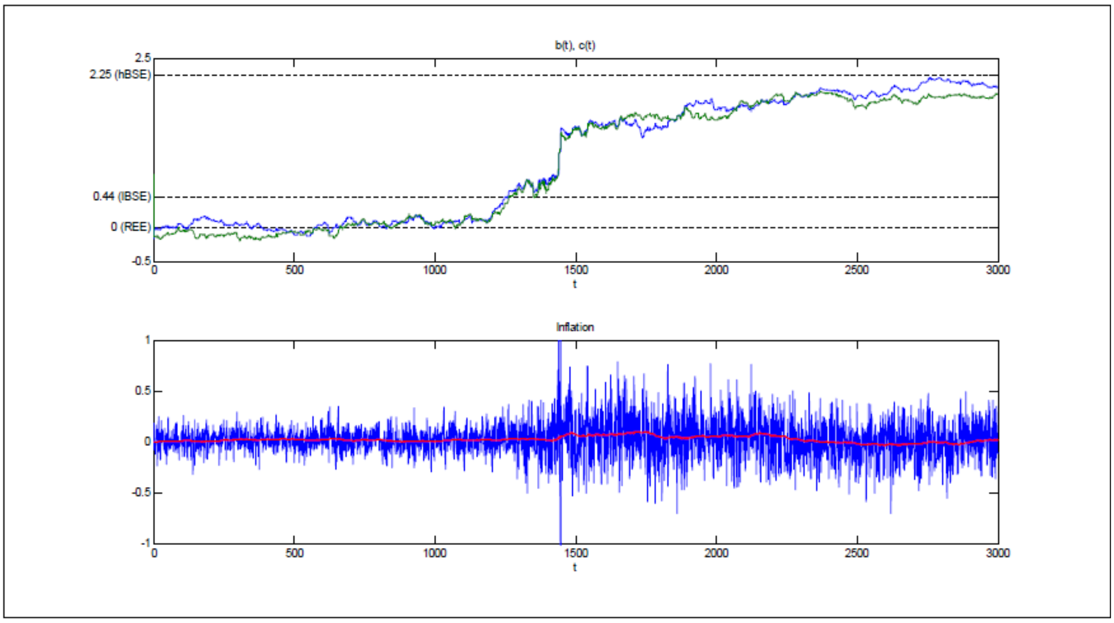


Figure 8: From REE to hBSE ($\beta = 0.8$, $\gamma = 0.28$, $\rho_v = 0.3$). Line c). in figure 3.

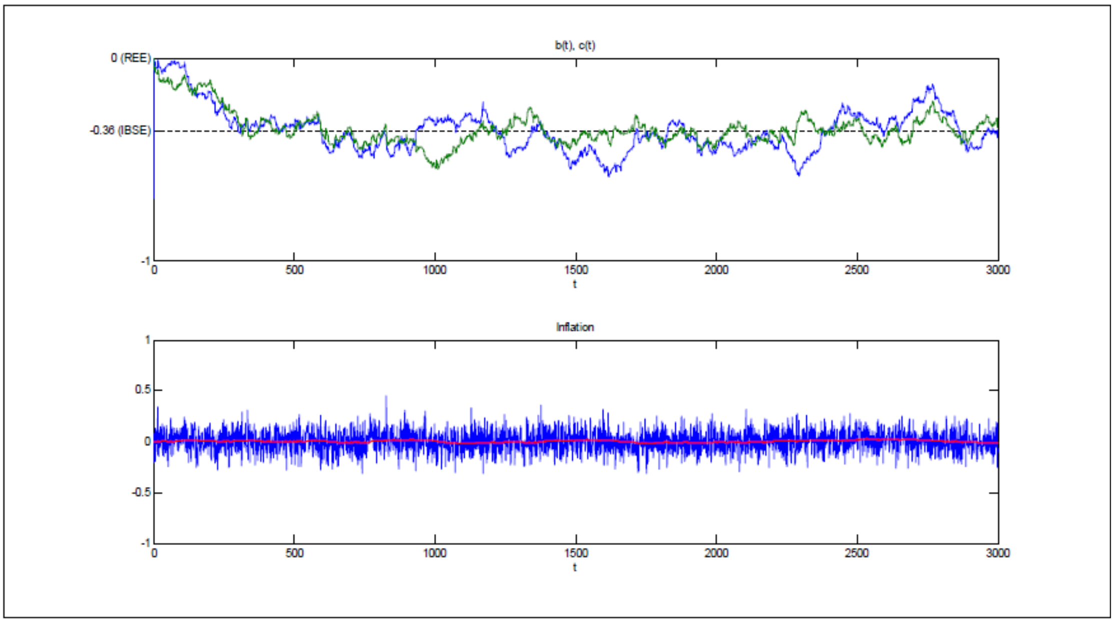


Figure 9: Convergence to lBSE ($\beta = 0.8$, $\gamma = 0.7$, $\rho_v = 0.8$). Line d). in figure 3.

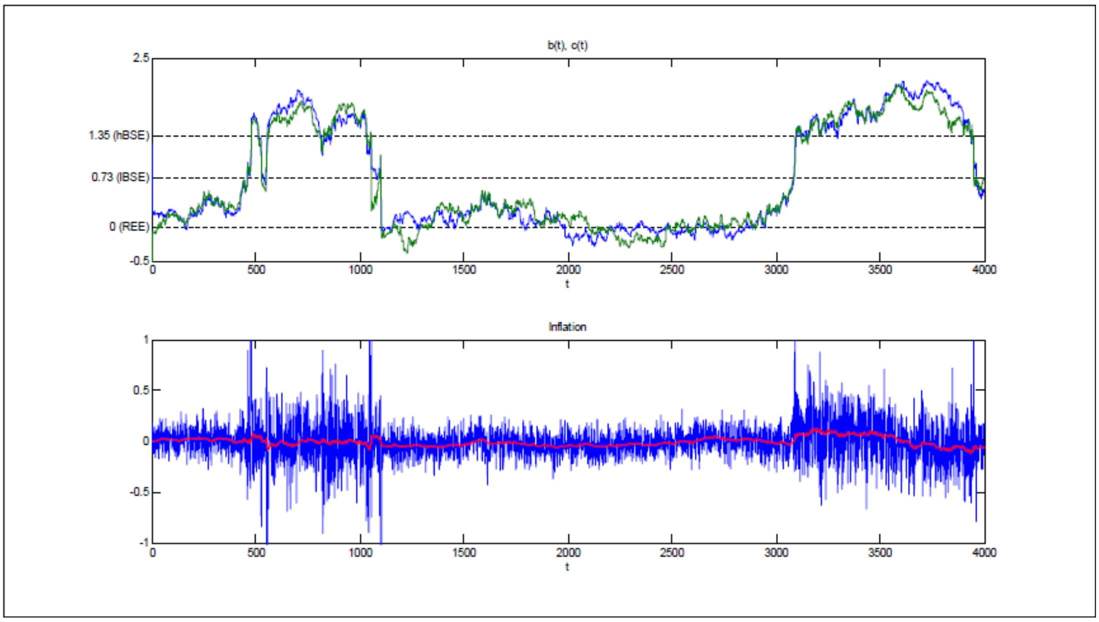


Figure 10: From REE to hBSE and back two times ($\beta = 0.8$, $\gamma = 0.21$, $\rho_v = 0.4$).

estimates dynamics more volatile and hence making jumps more likely.

Numerical simulation shows that dynamics similar to the latter can be generated considering more than two institutional forecasters with less extreme calibration[7]. Analytical results for such cases require a quite cumbersome computational analysis that is far beyond the scope of this work. This will be object of future investigation.

## 6 Conclusion

Adaptive learning in macroeconomics has been always presented as a bounded rationality approach since a central hypothesis is that agents don't recognize the self-referential nature of the model. In other words, agents focus only on exogenous determinants of the economy by-passing all issues linked to interactions among them. This feature results as an ad-hoc departure from full rationality paradigm and, as such, it weakens the theoretic robustness of this approach. More importantly the bounded rationality hypothesis prevents the explicit modelling of interdependence between agents' expectations that is widely recognized to be responsible for crisis triggering. Gaballo (2009) shows how to extend the approach to deal with such issues. Here we have used such results to model endogenous changes in volatility regimes due to emergence of interdependence among agents' expectations. We have also shown a simple way to reconcile the standard use of adaptive learning approach with the idea agents recognize the self-referential nature of the economy.

We have investigated a simple Lucas-type monetary model in which inflation depends on expectation of current inflation and other exogenous determinants. In this setting we assumed

[7]The programm is available upon request to the author.

expectations are interdependent in two respect. Firstly private sector relies evenly on two institutional forecasters. The latter are the only ones among agents having resource to gather and analyse efficiently information. In fact, each institutional forecaster implements statistical techniques to learn in real time the rational expectation, that is the fundamental inflation. The second way by which expectations are interdependent is due to behavioral uncertainty hypothesis. Behavioral uncertainty means that each institutional forecaster doesn't have perfect information about the other one's simultaneous expectations, but only a noisy signal of it. Given non-negligibility of institutional forecasters' expectations, they do have incentive to condition their expectations to these noisy signals in order to minimize their forecast error variance. In particular, they have to understand whether or not actual deviations from the esteemed fundamental rate are due to idiosyncratic departure of others rationality from the rational one. In sum, institutional forecasters have to learn not only about the fundamental inflation rate (as in standard adaptive learning literature) but also about rationality of others.

We have proved how the interaction of these two channels of expectations interdependence and constant gain adaptive learning can give rise to two type of learneable equilibria, namely the rational expectation equilibrium and a behavioral sunspot equilibrium. The former occurs whenever both agents' estimates of the optimal weight for noisy behavioral observation converge in distribution to zero, the latter arises otherwise. BSE are equilibria in which volatility of behavioral noisy observations enters in the actual law of motion generating excess inflation volatility. More importantly, constant gain learning generates endogenous, unpredictable and persistent switches in volatility regimes. These changes are obtained without any aggregate shock exogenously imposed. On the contrary excess volatility is triggered by noise justified by behavioral uncertainty at a micro level. The model has the advantage of being perfectly consistent with REE behavior and, nevertheless, it has the potentiality to exhibit endogenous structural changes.

## References


[1] Adam K.,Marcet A., Nicolini J.P. (2008a) "Stock Market Volatility and Learning". *ECB Working Paper Series* no. 862.

[2] Adam K.,Marcet A., Nicolini J.P. (2008b) "Internal Rationality and Asset Prices". *mimeo.*

[3] Allen F., Morris S., and Shin S.H. (2006) "Beauty Contests and Iterated Expectations in Asset Markets", *Review of Financial Studies.* Vol. 19, 719-752.

[4] Amato J.D., Morris, S. and Shin, H.S. (2005) "Communication and Monetary Policy". *BIS Working Paper* No. 123. Available at SSRN: http://ssrn.com/abstract=845444

[5] Branch W.A. and Evans G.W (2006) "Intrinsic Heterogeneity in Expectations Formation". *Journal of Economic Theory.* Vol. 127, 264-295.

[6] Branch W.A. and Evans G.W. (2007) "Endogenous Volatility and Model Uncertainty" *Review of Economic Dynamics.* Vol. 10, 207-237.

[7] Brennan, M. J. and Xia Y., (2001) "Stock Price Volatility and Equity Premium". *Journal of Monetary Economics.* Vol. 47, 249-283.

[8] Brock W. A. and Hommes C. H. (1997) "A rational route to randomness." *Econometrica* Vol. 65, 1059-1095.

[9] Brock W. A. and Hommes C. H. (1998) "Heterogeneous beliefs and routes to chaos in a simple asset pricing model." *Journal of Economic Dynamics and Control* Vol. 22, 1235-1274.

[10] Bullard J., Evans G. W., and Honkapohja S. (2005) "Near-Rational Exuberance". *ECB Working Paper* No. 555;

[11] Carceles-Poveda E. and Giannitsarou C. (2008) "Asset Pricing with Adaptive Learning" *Review of Economics Dynamics* Vol. 11, 629-651.

[12] Cogley, T.W. and Sargent, T.J., (2005a). "Drifts and volatilities: Monetary policies and outcomes in the post WWII US." *Review of Economic Dynamics* Vol. 8, 262–302.

[13] Cogley T. and Sargent T. J., (2005b) "The Conquest of U.S. Inflation: Learning and Robustness to Model Uncertainty" ECB Working Paper No. 478. Available at SSRN: http://ssrn.com/abstract=701291

[14] Cogley T.and Sargent T. J., (2008) "The market price of risk and the equity premium: A legacy of the Great Depression?" *Journal of Monetary Economics* Vol.55, 454-476.

[15] Evans G.W. Honkapohja S. (2001) “Learning and Expectations in Macroeconomics”. Princeton, Princeton University Press.

[16] Evans G.W. Honkapohja S. Williams N. (2005) "Generalized Stochastic Gradient Learning". CESifo Working Paper No. 1576.

[17] Evans G.W. and Ramey G. (2006). "Adaptive expectations, underparameterization and the Lucas critique" *Journal of Monetary Economics* Vol. 53, 249–264.

[18] Gaballo G. (2009) "Assessing others' rationality in real time" *Ph.D. dissertation, University of Siena, mimeo.*

[19] Giannitsarou C. (2003) "Heterogeneous Learning", *Review of Economic Dynamics*, Vol. 6, 885-906.

[20] Guse E. (2005) "Stability properties for learning with heterogeneous expectations and multiple equilibria", *Journal of Economic Dynamics and Control*, Vol. 29, 1623-1642.

[21] Kasa K. (2000) "Forecasting the forecast of others in frequency domain", *Review of Economic Dynamics*, Vol. 3, 726-756.

[22] Kydland F.E. and Prescott E.C., (1977) "Rules rather than discretion: The inconsistency of optimal plans". *Journal of Political Economy* Vol. 85, 473–491.

[23] Lucas R., (1973) " Some International Evidence on Output-Inflation Tradeoffs". *American Economic Review*. Vol. 63, 326-34.

[24] Marcet A. Sargent T. (1989) "Convergence of Least-Square Learning in Environments with Hidden State Variables and Private Information". *The Journal of Political Economy*, Vol. 97, No. 6, 1306-1322.

[25] Morris S. and Shin H.S. (2001) "Unique equilibrium in a model of self-fulfilling currency attack" in *New Research in Financial Markets,* edited by Biais B. Pagano M. Oxford, Oxford Press.

[26] Morris S. and Shin H.S. (2002) "Social Value of Public Information". *American Economic Review*, Vol.95, issue 5, 1521-1534.

[27] Pearlman and Sargent (2004) "Knowing the forecast of others", *Review of Economic Dynamics,* Vol 8, 480-497.

[28] Primiceri, G. (2005) "Time Varying Structural Vector Autoregressions and Monetary Policy". *Review of Economic Studies* Vol.72, 821-852.

[29] Primiceri, G. (2006) "Why Inflation Rose and Fell: Policy-Makers' Beliefs and U. S. Postwar Stabilization Policy" *Quarterly Journal of Economics*, Vol. 121, 867-901.

[30] Sargent T.J. (1987). "Macroeconomic Theory", second ed. Academic Press, New York.

[31] Sargent T.J. (1991). "Equilibrium with signal extraction from endogenous variables". *Journal of Economic Dynamics and Control*, Vol. 15, Issue 2, 245-273.

[32] Sargent T.J. (1999). "The Conquest of American Inflation" Princeton Univ. Press, Princeton, NJ.

[33] Sims, C.A. and Zha T. (2006). "Were there regime switches in US monetary policy?" *American Economic Review* Vol. 96, 54–81.

[34] Singleton K.J. (1987). "Asset prices in a time-series model with disparately informed competitive traders." In: W.A. Barnett and K.J. Singleton, Editors, *New Approaches to Monetary Economics*, Cambridge Univ. Press.

[35] Timmermann, A., (1994) "How learning in financial markets generates excess volatility and predictability in stock prices" *Quarterly Journal of Economics* Vol. 108, 1135–1145.

[36] Timmermann, A., (1996) "Excess volatility and predictability of stock prices in autoregressive dividend models with learning" *Review of Economic Studies* Vol. 63, 523–557.

[37] Townsend R.M. (1983) "Forecasting the forecast of others" *The Journal of Political Economy*, vol. 91, 546-588.

[38] Walsh C., (2003). "Monetary Theory and Policy" second ed. MIT Press, Cambridge, MA.

[39] Woodford M., (2003). "Interest and Prices". Princeton Univ. Press, Princeton, NJ.