Bifurcation Scenarios in a Heterogeneous Agent, Multiplier-Accelerator Model

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Abstract

Samuelson's linear multiplier-accelerator model of the business cycle is modified in order to explore the influence of heterogeneous agents' expectations on fluctuations resulting from the combined effects of the multiplier and acclerator principles. The expected income is a nonlinear mix of extrapolative and reverting expectation rules based on previous period income, which are used to predict economic activity. The equilibrium of Samuelson's model is also a fixed point of the extended model, but other limit sets exist, including sequences of points on an invariant closed curve with periodic and quasiperiodic behavior, chaotic attractors and fixed points above and below the Samuelsonian equilibrium. General bifurcation scenarios over relevant parameter ranges are described.

Mathematics Subject Classification: 03H10, 37E99

1 Introduction

In this paper we investigate how a simple version of heterogeneous agents' expectations modifies the basic dynamical structure of the multiplier-accelerator model due to Samuelson (1939). Consumption depends on the expected value of present income rather than lagged income. Agents' expectations are of two types: one ensures a destabilizing force for values near the equilibrium, but the economy neither explodes nor contracts indefinitely due to a global stabilizing expectation that dominates when the economy deviates too much from its equilibrium. National income is determined as a nonlinear mix of extrapolative and reverting expectations formation rules (prototypical predictors used in recent literature on financial markets). The total level of economic activity depends endogenously on the proportion of agents using the predictors. The interacting expectations permit a greater variety of attracting sets including point equilibria above and below the (unique) Samuelsonian equilibrium and closed curves on which lie both quasiperiodic and periodic cycles, as well as chaotic attractors over a small area of the parameter subspace.

The remainder of the paper is organized as follows. Samuelson's original business cycle model, new hypotheses regarding expectations formation and aggregation rules and the resulting revised model are presented in Section 2. In section 3 the local properties of the model are analyzed through the linear approximation. In sections 4 and 5 we use analysis and numerical simulations to study typical bifurcation scenarios and the global properties of the model. In section 6 conclusions are offered.

2 The model

The model proposed by Samuelson in 1939 incorporates the Keynesian multiplier, a multiplicative factor that relates expenditures to national income and the accelerator principle whereby induced investment is proportional to increases in consumption. An increase in investment leads to an increase in national income and consumption (via the multiplier effect) which in turn raises investment (via the accelerator process). This feedback mechanism may generate an oscillatory behavior of output.

Consumption in period t is hypothesized as a constant proportion of national income in period $t-1$

$$C_t = bY_{t-1} \quad 0 < b < 1$$

where b is the propensity to consume out of previous period income. Investment is assumed partly autonomous and independent of the business cycle, denoted I_a, and partly induced, proportional to changes in consumption with acceleration coefficient, k:

$$I_t = I_a + k(C_t - C_{t-1}) \quad k > 0. \tag{1}$$

Combining these with the equilibrium condition for a closed economy (output is fully utilized as either consumption or investment goods) $Y_t = C_t + I_t$, we obtain a second-order linear difference equation, in the income variable:

$$Y_t = I_a + b(1+k)Y_{t-1} - bkY_{t-2}. \tag{2}$$

The fixed point of (2), the long-run equilibrium output, is determined as

$$\mathcal{Y} = \frac{1}{1-b} I_a. \tag{3}$$

Stability of the fixed point requires $b < \frac{1}{k}$. Damped oscillations occur only in the area with $b < 1/k$ and $b < 4k/(1+k)^2$. In that case temporary business cycles arise due to the interplay of the multiplier and the accelerator however, changes in economic activity either die out or explode (persistent cycles only occur for a nongeneric boundary case).

We next extend the model to include expectations. It is assumed that consumption depends on the expected value of current income, conditional on the previous period income

$$C_t = bE_{t-1}[Y_t].$$

The aggregate expectation $E_{t-1}[Y_t]$ is formed as a weighted average of extrapolative (denoted 1) and reverting (denoted 2) expectations:

$$E_{t-1}[Y_t] = w_t E^1_{t-1}[Y_t] + (1-w_t)E^2_{t-1}[Y_t] \quad 0 < w < 1.$$

Expectations are formed with reference to a “long-run” equilibrium which is taken to be the fixed point of Samuelson’s linear model, $\mathcal{Y}$. In the extrapolative expectation, or trend, formation rule, agents either optimistically believe in a boom or pessimistically expect a downturn. Such expectations are formalized as

$$E^1_{t-1}[Y_t] = Y_{t-1} + \mu_1(Y_{t-1} - \mathcal{Y}) \quad \mu_1 > 0.$$

If output is above (below) its long-run equilibrium value, $\mathcal{Y}$, people think that the economy is in a prosperous (depressed) state and thus predict that national income will remain high (low).

Equilibrium-reverting expectations are formed as

$$E^2_{t-1}[Y_t] = Y_{t-1} + \mu_2(\mathcal{Y} - Y_{t-1}) \quad 0 < \mu_2 < 1$$

where μ_2 captures the agents’ expected adjustment speed of the output towards its long-run equilibrium value.

The more the economy deviates from $\mathcal{Y}$, the less weight the agents put on extrapolative expectations, that is, agents believe that extreme economic conditions are not sustainable. Formally, the relative impact of the extrapolative rule depends on the deviation of income from equilibrium at the time that expectations are formed:

$$w_t = \frac{1}{1 + \left(\gamma\left(\frac{Y_{t-1} - \mathcal{Y}}{\mathcal{Y}}\right)\right)^2} \qquad \gamma > 0$$

with γ a scale factor. The percentage gap is typically less than one which, when squared, results in a small number. Setting $\gamma > 1$ increases the weight factor, resulting in a more realistic distribution between extrapolative and equilibrium-reverting expectations. If $\gamma = 10$ and the percentage gap is 10%, the proportion of agents using E^1 is 50%; the proportion is 99% for $\gamma = 1$. Close to equilibrium the trend-following expectation dominates (and at $Y_t = \mathcal{Y}$, $w_t = 1$, acting as a destabilizing force for any small deviation from the long-run equilibrium). At large distances from $\mathcal{Y}$ the reverting expectation dominates, acting as a global stabilizing force.

Substituting in expectations gives a second-order nonlinear difference equation $Y_t = f(Y_{t-1}, Y_{t-2})$ which is converted to a first-order system (Y_t, Z_t) using an auxiliary variable, $Z_t = Y_{t-1}$

$$\begin{aligned} Y_t &= I_a + b(1+k)E_{t-1}[Y_t] - bkE_{t-2}[Z_t] \\ Z_t &= Y_{t-1} \end{aligned} \tag{4}$$

with Jacobian matrix

$$J(Y,Z) = \begin{pmatrix} b(1+k)\frac{dE_{t-1}[Y_t]}{dY_{t-1}} & -bk\frac{dE_{t-2}[Z_t]}{dZ_{t-1}} \\ 1 & 0 \end{pmatrix}.$$

3 Local dynamics

Consider the local stability of the fixed points of the system represented in (4). The equilibrium value for Samuelson's model $\mathcal{Y}$ is also a fixed point of the modified model, at which trend followers are predicting perfectly, $w = 1$ and the Jacobian simplifies to:

$$J(\mathcal{Y}) = \begin{pmatrix} b(1+k)(1+\mu_1) & -bk(1+\mu_1) \\ 1 & 0 \end{pmatrix}$$

with trace tr$J = b(1+k)(1+\mu_1)$ and determinant det$J = bk(1+\mu_1)$. Given the stability conditions for a two-dimensional system:

$$\begin{aligned} 1 + \text{ tr } J(\mathcal{Y}) + \text{ det } J(\mathcal{Y}) &> 0 \quad (i) \\ 1 - \text{ tr } J(\mathcal{Y}) + \text{ det } J(\mathcal{Y}) &> 0 \quad (ii) \\ 1 - \text{ det } J(\mathcal{Y}) &> 0. \quad (iii) \end{aligned}$$

we have the first condition always true, eliminating the possibility of flip bifurcations. The second and third conditions, respectively:

$$\mu_1 < \frac{1-b}{b} \quad \text{and} \quad \mu_1 < \frac{1-kb}{kb} \tag{5}$$

are not necessarily satisfied, leaving open the possibility of both fold (specifically pitchfork) and Neimark-Sacker bifurcations (henceforth referred to as simply Neimark bifurcations). The parameter assumptions are simply that μ_1, $k > 0$ and the binding inequality is condition (ii) if $k < 1$, condition (iii) if $k > 1$.

In a discrete-time setting investment is represented by the difference in the capital stock ΔK_t. Substituting in (1) the hypothesis that consumption is a constant proportion of income gives $\Delta K / \Delta Y = kb$. This ratio gives an indication of the capital/output ratio which is definitely greater than 1, suggesting that, with $b \leq 1$, $k \geq 1$. In a normal setting then, the critical value for the pitchfork bifurcation is surpassed. The quadratic term in the weight, determining the agent ratio, implies the potential existence of 2 real fixed points symmetric about $\mathcal{Y}$, given by

$$(\bar{Y} - \mathcal{Y})^2 = \frac{\mathcal{Y}^2\big(b(1+\mu_1) - 1\big)}{\gamma^2\big(b(\mu_2 - 1) + 1\big)}.$$

They are complex-valued for $\mu_1 < (1-b)/b$, become real and equal in value to $\mathcal{Y}$ at the critical value $\mu_1 = (1-b)/b$. For $\mu_1 > (1-b)/b$ they are two positive, real equilibria, one larger and one smaller than $\mathcal{Y}$, respectively $\bar{Y}_1$, $\bar{Y}_2$. These and their Neimark bifurcations, in combination with the stable and unstable manifolds of the saddle node at $\mathcal{Y}$ result in some interesting dynamics which we explore below.

4 Bifurcation scenario 1

In this section the bifurcation parameter is chosen to be μ_1, the coefficient used by trend followers in their expectations formation. (A comparison of dynamics

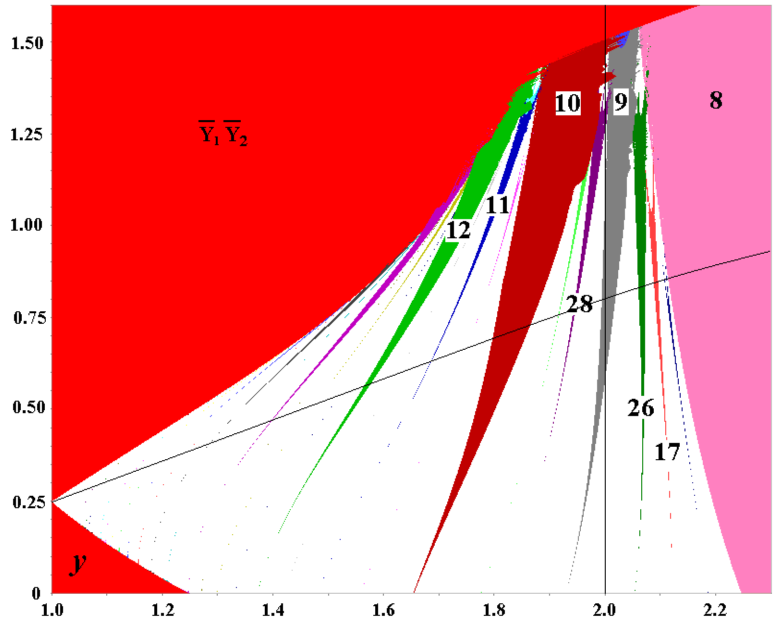

Figure 1: Parameter space (k, μ_1)

between the original and revised models over the original parameter space (k, b) is given in Lines and Westerhoff, forthcoming). Recall that μ_1 appears in the stability conditions for $\mathcal{Y}$, (6), and is only restricted to positive values. That is, a period with income higher (lower) than Samuelson's equilibrium is expected to be followed by another period with income above (below) $\mathcal{Y}$. For instance, if $\mu_1 = 1$ the optimistic trend-follower believes that national income will be above what it was in the previous period by an amount equal to the distance of the previous period from $\mathcal{Y}$. In Figure 1 the double-parameter bifurcation diagram is provided over $k \in (1, 2.3)$, $\mu_1 \in (0, 1.6)$, using a standard constellation of parameters: $b = 0.8$ giving $\mathcal{Y} = 5000$, $\mu_2 = 0.5$, $\gamma = 10$, $I_a = 1000$, $Y_0 = 3500$, $Z_0 = 3500$. For the algorithm, infinity is 10^{10}, transients are 5000 with maximal period 28 and precision epsilon set at 0.1. This and all following plots were produced with the open-source software iDMC - Copyright Marji Lines and Alfredo Medio, available at www.dss.uniud.it/nonlinear.

The parameter space represents roughly three zones. The areas in red on the left represents stable fixed points, below the critical value of the pitchfork bifurcation, $\mu_1 = 0.25$, the attractor is the Samuelsonian equilibrium, above it trajectories are attracted to either $\bar{Y}_1$ or $\bar{Y}_2$, depending on the initial values. Moving right, the background white area is characterized by quasi-periodic, higher-order periodic fluctuations or chaotic attractors while the colored Arnol'd tongues represent stable periodic cycles. The curve emanating from $\mu_1 = 0.25$ represents the critical values of the (subcritical) Neimark bifurcations of these 2 fixed points, there can be no stable $\bar{Y}_1$ or $\bar{Y}_2$ for parameter combinations to the right of the curve. Moving again right, the pink region represents a period-8 cycle that appears to be (and is in fact) different than the periodic cycles on the invariant curve. High values of the accelerator lead to instability as entrepreneurs invest heavily in order to keep up with the increased demand of goods and services created by increases in consumption. The amplitude of fluctuations for values of $k > 2.3$ is unrealistic. An economy experiencing such cycles would generate different relations among macrovariables, expectations would change and the model would no longer be valid.

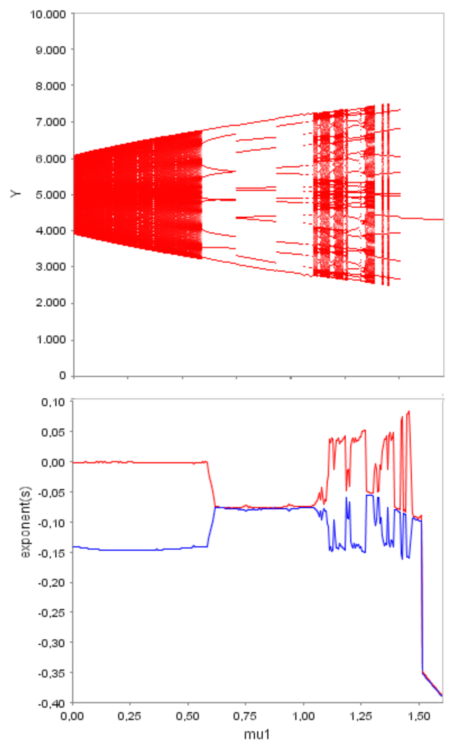

Figure 2: Scenario 1: diagram above; Lyapunov exponents below.

For the sake of clarity we consider two bifurcation scenarios, choosing realistic values for one parameter and allowing the other to vary over the range depicted in the double parameter bifurcation diagram in Figure 1. In this section we set $k = 2.0$ and vary μ_1. At this value for the acceleration coefficient the binding stability condition for $\mathcal{Y}$ is the critical value of the Neimark bifurcation $\mu_1 = -0.375$. Beginning at $k = 2$, $\mu_1 = 0$ in Figure 1, that value (of the supercritical Neimark bifurcation) has been surpassed and the limit sets are quasiperiodic attractors on an invariant curve. Attractors on the invariant curve continue until it loses stability and fixed points become the attractors. More precisely, at $\mu_1 = 0.25$, $\bar{Y}_1$ and $\bar{Y}_2$ are real in value but are unstable nodes, by 0.33 they are unstable focuses, and finally, by 0.79 they are stable focuses at $\bar{Y}_1 \approx 4576$, $\bar{Y}_2 \approx 5424$. The initial conditions used for Figure 1, (3500, 3500), fall into one of their basins from about $\mu_1 = 1.5$ through to the last value depicted, 1.6.

Bifurcation scenario 1 is presented in Figure 2: the single parameter bifurcation diagram for $\mu_1 \in (0, 1.6)$, above (using a transparency factor of 0.2 to get a better idea of the invariant distribution); the Lyapunov exponents over the same interval, below. These figures suggest that there are represented attractors of every type for trajectories beginning at (3500, 3500).

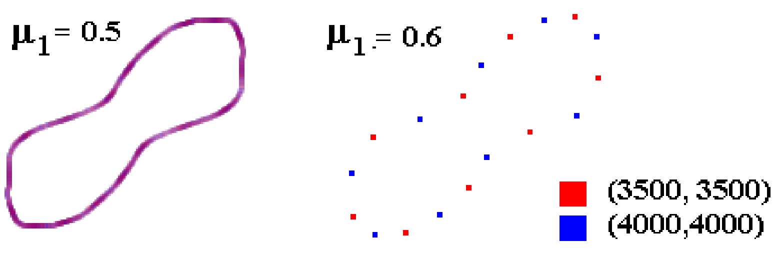

Figure 3: From quasiperiodic attractors to period-9 cycles.

For small values of the reaction coefficient the economy experiences fluctuations that are, for the most part quasiperiodic with the largest Lyapunov exponent (in red) at 0 and the second Lyapunov exponent (in blue) negative. (Exponent value have been connected for ease in viewing.) When the parameter value reaches the Arnol'd tongue for period-9 cycles, for these initial conditions around $\mu_1 = 0.6$, the exponents converge to a negative value. Then after leaving the 9-cycle tongue, around $\mu_1 = 1.15$, chaotic and periodic windows alternate until finally, near $\mu_1 = 1.5$ the trajectories are attracted to the fixed point $\bar{Y}_1$ which is strongly attracting. The scenario from different initial values would see even large shifting of critical values, shortening or extending the intervals for different types of limit sets, but the overall picture would be the same.

The parameter intervals around the change values are interesting. Consider around the first change from the quasiperiodic attractors on the invariant curve to the period-9 cycle. In Figure 3 two trajectories are present on the right and left sides, one with initial conditions the same as in the double parameter bifurcation diagram in Figure 1 (3500, 3500) with limit sets in red, the other with initial conditions closer to $\bar{Y}_1$ at (4000, 4000) and limit sets in blue. Just before entering the period-9 tongue, $\mu_1 = 0.5$, both trajectories converge to the invariant circle. Once inside the tongue, $\mu_1 = 0.6$, trajectories converge to period-9 cycles which are lying on the invariant curve, but they are distinct.

In Figure 4 the state space is presented with both axes from 0 to 10000. The periodic points of the distinct period-9 cycles typically lie close to the basin boundaries, one in blue and the other in green. It is interesting to note the shadowy imprint of the nearby period-8 cycle in the state space, which appears to be almost partitioned by 4 curves.

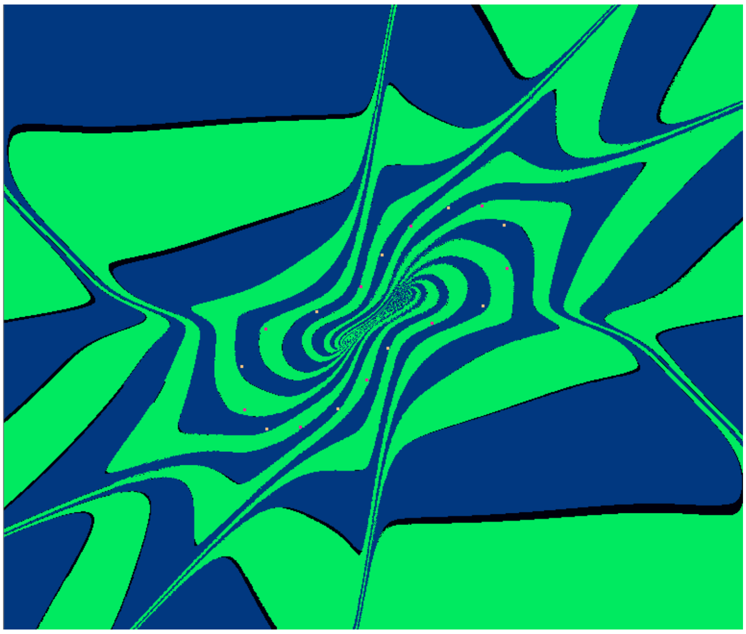

Figure 4: Basins for period-9 cycles at $\mu_1 = 0.6$.

The existence of 2 distinct period-9 cycles continues until the parameter reaches the end of tongue at around $\mu_1 = 1.1$. The limit sets change quite quickly over this interval for the parameter as can be observed in Figure 5, left, close ups of the bifurcation diagram and Lyapunov exponents in Figure 2, over $\mu_1 \in (1, 1.6)$ with the zero exponent line added for clarity. It would appear that over the interval $\mu_1 \in (1.1, 1.6)$ chaotic attractors alternate with periodic windows. The stable cycles include the period-9, which dominates until around the first substantial subinterval of chaotic attractors and the period-10 which is interspersed with chaotic attractors near the end of the interval. In Figure 5, right, a close up of the double bifurcation diagram suggests how complicated the basin structure becomes as the parameter gets closer to the value at which trajectories from (3500, 3500) are attracted to the fixed point. The rotation numbers indicate the periodicity (denominator) and number of rotations required for a full cycle (numerator). For example the purple area represents sequences that after three rotations on the invariant curve, complete a periodic cycle of 28 points.

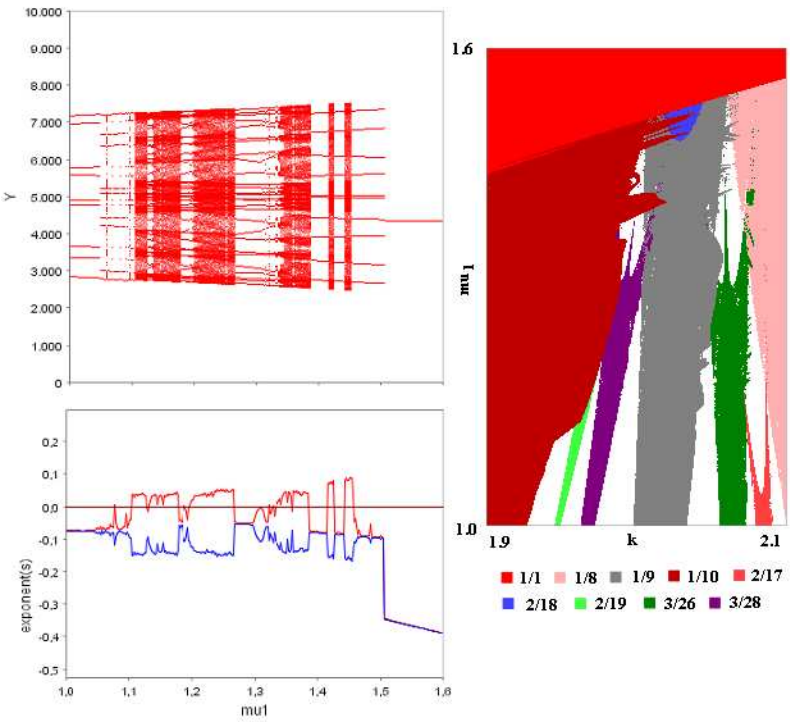

Figure 5: Close-ups diagram, exponents left; double bifurcation right.

The intricate basin structure is presented in Figure 6 (both axes from 2000 to 8000) for $\mu_1 = 1.15$, using 200 transients, 1000 iterations and 100 trials to find all of the limit sets. There are again 2 stable period-9 cycles with basins of attraction in green and cobalt blue, but these attract only initial conditions outside of the pink ring, and not all of those. The periodic points of the cycles are (barely) visible, lying close to the chaotic attractor in pink, whose basin is in dark blue. The basin of attraction of the strange attractor includes some external points and then extends over the inside of the curve, with the exception of the basins of the stable $\bar{Y}_1$ and $\bar{Y}_2$, visibly nested within their basins. From the initial point used in Figure 5, left, the trajectory converges to the chaotic attractor. Again, notice that the area outside of the chaotic ring appears to be partitioned into 8 pieces.

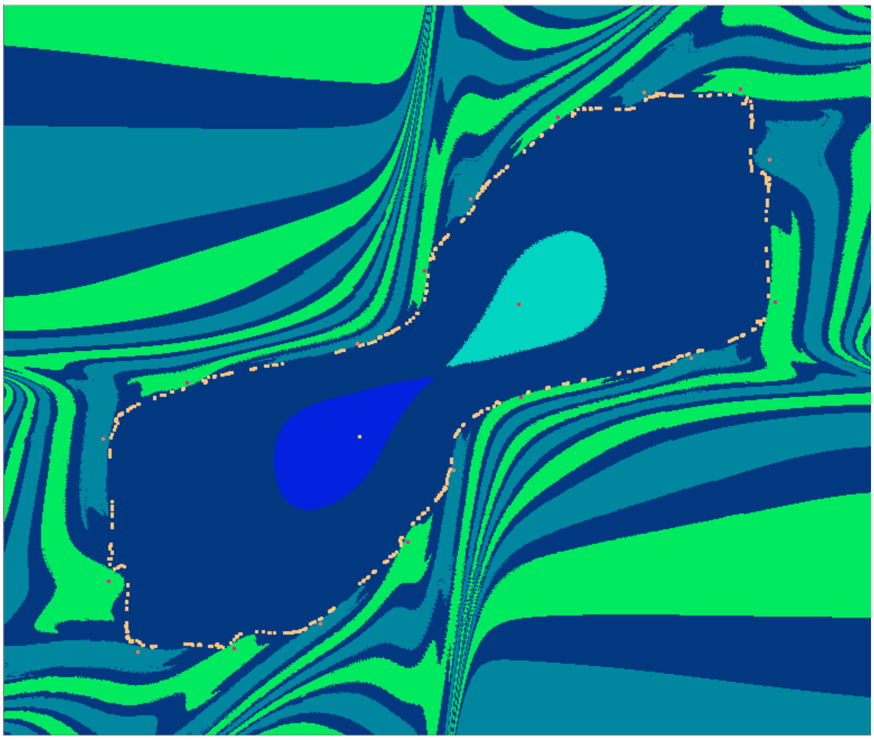

Figure 6: Fixed points, period-9 cycles and a chaotic attractor for $\mu_1 = 1.15$.

In Figure 7, left we vary the initial conditions, using the same standard parameter constellation and $\mu_1 = 1.15$, to simulate the different limit sets ($\bar{Y}_1$ attracting for initial condition (4700, 4700), for example). The period-9 cycle, which attracts initial condition (2700, 2700) is resting close to the chaotic attractor to which the trajectory from (3700, 3700) converges. The strange attractor lies on the chaotic ring and the accumulating of points in particular areas, close to the periodic points of this and the second period-9 cycles, creates "frills" on the ring where trajectories get waylaid before moving on. The homoclinic connection of the manifolds of $\mathcal{Y}$ occurs for a larger value of the parameter, for $\mu_1 = 1.15$ the equilibrium of Samuelson's model lies in the basin of the chaotic attractor. In Figure 7, right, are the first 20 iterations of its unstable manifolds using length 1.

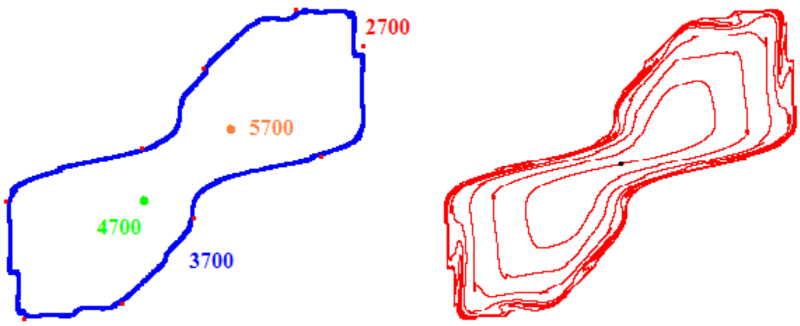

Figure 7: Limit sets of trajectories from given initial values.

As we increase the parameter the basins of the fixed points get larger, the chaotic attractor shrinks and the cycles of different periods come and go. In Figure 8 we have basins for $\mu_1 = 1.5$, axes over (0, 10000). It is clear that the homoclinic connection has already occurred as the basins of $\bar{Y}_1$ and $\bar{Y}_2$ are attached near the saddle point and the unstable manifolds of $\mathcal{Y}$ lead to these fixed points. Outside of the chaotic ring there are 2 period-9 cycles with basins in shades of blue a period-10 cycle with green basin.

5 Bifurcation scenario 2

The limit sets studied in scenario 2 likewise appear in scenario 1, but in a different order. Refer again to the double parameter bifurcation diagram in Figure 1. We now take the coefficient of acceleration k as the bifurcation parameter and set the reaction coefficient of the trend-followers expectations equal to a constant. For values of $\mu_1 > 0.25$ that is, for which the Samuelsonian equilibrium is unstable and the symmetric pair of fixed points around it are real, the scenario is qualitatively similar to that in Figure 9, for which $\mu_1 = 0.6$.

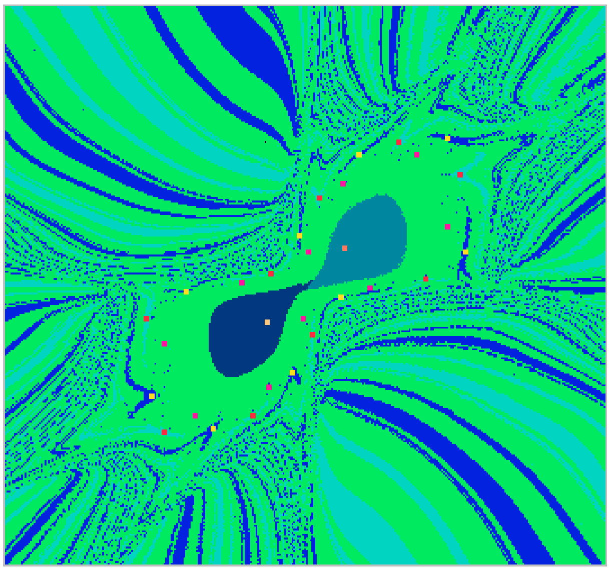

Figure 8: Basins of attraction $\mu_1 = 1.5$.

There are three general types of attractors over the scenario: fixed points, quasiperiodic and periodic attractors on an invariant curve and finally, a period-8 cycle (the extreme pairs of periodic points are very close to each other). The values of the fixed points do not depend on k, $\bar{Y}_1 = 4658$, $\bar{Y}_2 = 5342$, but of course, the eigenvalues and therefore the local dynamics do.

For low values both fixed points are stable, but with uncomplicated basins separated by the stable manifold of the saddle point $\mathcal{Y}$. As k increases the respective basins of attraction become ever more entwined, so that with fixed initial values, trajectories are captured by first one and then the other fixed point (see Figure 10, above left, for $k = 1.35$). This is observable in the switching behavior of the bifurcation diagram which increases in frequency the closer the parameter comes to the critical value of the global bifurcation, here around 1.36.

Notice that the change from a fixed point to the invariant curve comes long before reaching the critical value for the Neimark bifurcation of $\bar{Y}_1$, $\bar{Y}_2$ at approximately $k_c = 1.6176$ (with $\partial\lambda(k_c)/\partial k > 0$ and powers of $\lambda^i(k_c)$, $i = 1, \ldots, 4$ complex.) This, along with the jump to a large radius curve suggests that the simultaneous bifurcations are subcritical, for which the invariant curves determined by the Neimark bifurcations are unstable and act as boundaries for the basins of at-

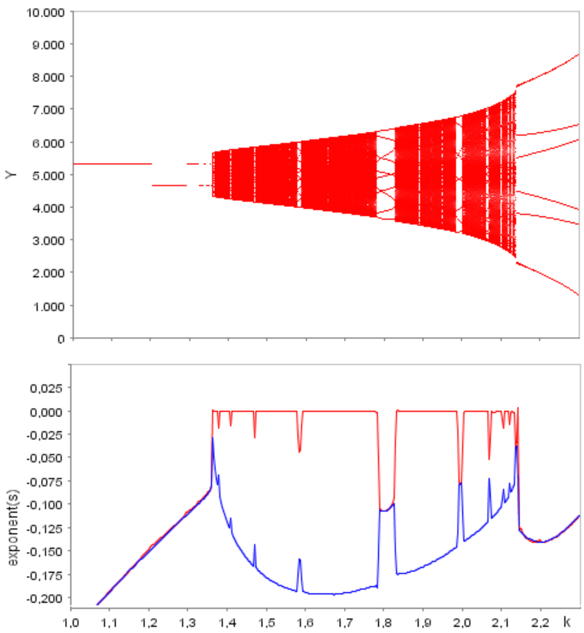

Figure 9: Scenario 2: diagram above; exponents below.

traction of the stable fixed points. In the upper-left part of Figure 10 the separatrix $w^s(\mathcal{Y})$ divides the state space into basins of attraction for $\bar{Y}_1, \bar{Y}_2$. The convolutions of the stable manifold form a ring of entwined basins around the three fixed points and, increasing k, an attracting invariant closed curve appears. At the creation of the attracting curve, call it Γ_s, a second curve, Γ_u, also appears which is enclosed in the first and repelling. The latter forms a separatrix, now between collections of initial conditions with trajectories tending to one or other of the stable foci and initial conditions with trajectories tending to the attracting Γ, see Figure 10, upper right. At this value there is a double homoclinic loop, that is, the basins are disjoint and there has been a homoclinic bifurcation of $(\mathcal{Y})$ for a smaller k. Next, the subcritical Neimark bifurcations for $\bar{Y}_1$, $\bar{Y}_2$ occur, the basins disappear altogether and as k is further increased all attractors are on the invariant curve.

The process by which the attractor Γ_s comes to co-exist with the stable foci, and the separatrix Γ_u defining its basin of attraction is not entirely clear. The likely sequence leading to the formation of Γ_s is a saddle cycle, proposed in a number of works, see Agliari (forthcoming), Agliari, Gardini and Puu (2004) and Agliari and Dieci (forthcoming).

The last change occurs as the shadowy period 8 focus cycle suddenly appears to co-exist with Γ_s. Figure 10 below shows this sequence, at around $k = 2.115$ the periodic points appear around and near the closed curve, by $k = 2.215$ the only remaining attractor is the period-8 cycle. For values of k beyond those shown in Figure 9 the fluctuations lead to ever larger values (and negative values) and eventually it becomes unstable after which there are no attractors.

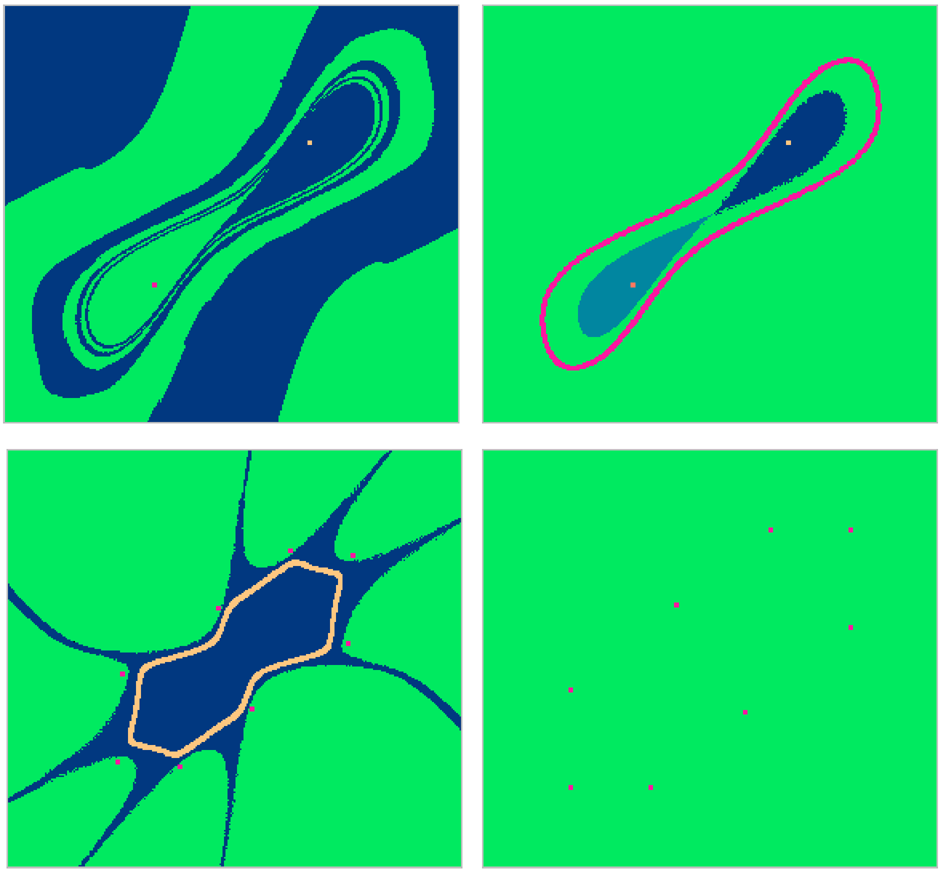

Figure 10: Basins of attraction.

The curve Γ_s is destroyed by a heteroclinic loop sequence (see above references). In Figure 10, lower left the periodic points and associated saddle points are very near to each other and lie on the boundaries of the basin of attraction for the focus cycle $B(C)$. The branches of the stable manifolds of the saddle cycle serve as separatrix between $B(C)$ and $B(\Gamma_s)$. The outer branch of the unstable manifold of the saddle leads to the focus cycle, the inner branch leads to the invariant curve. As k is increased, the inner unstable branch of the saddle point i becomes tangential to the inner stable branch of nearby saddle point j, and this happens all around the cycle. This heteroclinic tangency starts a tangle, followed by a transversal crossing of these branches and another heteroclinic tangency. Transversal crossings are usually associated with chaotic repellers and long chaotic transients. A hint of this can be seen in Figure 9 as there is a slight rise in the Lyapunov characteristic exponent near the bifurcation interval. At the end of the tangle the branches are switched in position. The unstable branches of the saddle point i tend to the nearby stable foci (to the right and left, h and j) forming a heteroclinic saddle-focus connection that leaves no initial condition leading to Γ_s.

6 Conclusions

The linear multiplier-accelerator model introduced by Samuelson in 1939 has been a constant source of interest to economists. It is a classic example of a business cycle model based on the combined effects of the multiplier and accelerator principles. The dynamics are completely understood. Unfortunately, only very particular parameter values lead to persistent fluctuations, a drawback for a business-cycle model. On the other hand a simple alteration to the consumption hypothesis in the spirit of the heterogeneous agents' approach currently in vogue leads to a variety of recurrent dynamical behavior. Besides the fixed point of Samuelson's model and 2 other fixed points (one above and one below) there are periodic, quasiperiodic and (more rarely) chaotic attractors over much of the realistic subarea of the parameter space. In the paper two typical bifurcation scenarios have been described using analysis and numerical simulations.

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